Department of Mathematics
Comprehensive Examination
2024 Fall Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

1. Define

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{5^n n! z^n} + \sum_{n=0}^{\infty} \frac{z^n}{3n^2 + n!}$$

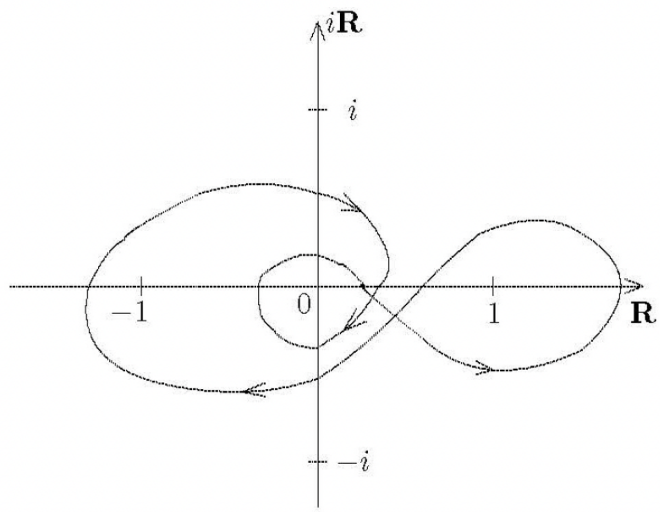
   Given that each series converges for all $z \neq 0$, find $\oint_C f(z)dz$ if C is the unit circle taken counterclockwise. Completely explain your answer.

2. Find all Laurent expansions of $f(z) = \frac{4}{z^2+2z-3}$ centered at $z_0 = 1$ and state their regions of convergence.

3. Find a polynomial $p(z) = az^2 + bz + c$ such that

$$\begin{aligned} \int_C \frac{p(z)}{z+1} dz &= 4\pi i, \\ \int_C \frac{p(z)}{z^2} dz &= 8\pi i, \text{and} \\ \int_C \frac{p(z)}{(z-1)^2} &= 12\pi i \end{aligned}$$

   where C is the closed contour illustrated.



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Math 660: Topology

No Topology Exam was given in Fall 2024

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Time: 2 hours

Math 675: Differential Equations

No Differential Equations Exam was given in Fall 2024

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Time: 2 hours

Math 680: Optimization

10. Prove or disprove, using the Complementary Slackness Theorem:
(0,2,8,18) is the optimal solution to the following linear programming problem.

$$\begin{array}{lrcrcrcrl}
\text{Minimize} & 8x_1 & - & 2x_2 & - & x_3 & - & 6x_4 & \\
\text{Subject to} & x_1 & + & x_2 & - & x_3 & + & x_4 & = 12 \\
 & -2x_1 & + & 3x_2 & & & + & 2x_4 & = 42 \\
 & x_1 & & & + & x_3 & + & x_4 & \leq 26 \\
 & & & & & & & x_1, x_2, x_3, x_4 & \geq 0
\end{array}$$

11. Anna bakes fresh muffins, scones and danish for a nearby bakery. The bakery has imposed a few nutritional requirements, based on their customers' needs. Chia seeds seem to be all the rage these days, so the bakery ask that the total amount of chia seeds be at least 56 ounces. Similarly, cranberries, for their high vitamin C content, are in high demand, and thus, the bakery asks for at least 22 ounces altogether, in the pastries. Along with these healthy guidelines, the bakery also asked that the amount of butter be limited to at most 21 cups.

The following table summarizes the requirements.

Pastry	**Amount of Chia Seeds (oz)**	**Amount of Cranberries (oz)**	**Amount of Butter (cups)**	**Cost per item**
Danish	5	1	0	$3
Scone	3	1	2	$2
Muffin	6	4	3	$6
Total	at least 56 oz.	at least 22 oz.	at most 21 cups	

Please formulate an appropriate Linear Programming problem to minimize costs for Anna, while satisfying the bakery's constraint. Solve the LP problem using the Simplex method and determine how many of each kind of pastry (Danish, scone, muffin) Anna should make.

12. Consider the following maximization problem with associated table showing the beginning and ending tableaux.

$$\begin{array}{lrrrl} \max & 4x_1 & +6x_2 & +3x_3 & \\ \text{s/t} & x_1 & +2x_2 & +\ x_3 & \le 42 \\ & 3x_1 & +3x_2 & +4x_3 & \ge 26 \\ & 2x_1 & +\ x_2 & & \le 30 \\ & & & x_1, x_2, x_3 & \ge 0 \end{array}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	2	1	1	0	0	42
a_1	3	3	4	0	-1	0	26
x_6	2	1	0	0	0	1	30
	-4	-6	-3	0	0	0	0
x_5	0	$\frac{9}{2}$	0	4	1	$-\frac{1}{2}$	127
x_3	0	$\frac{3}{2}$	1	1	0	$-\frac{1}{2}$	27
x_1	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	15
	0	$\frac{1}{2}$	0	3	0	$\frac{1}{2}$	141

Answer the following questions using techniques of sensitivity analysis. Treat each situation separately, as it refers to the original problem.

(a) How much can the coefficient c_1 of x_1 in the objective function such that the optimal feasible solution of $(x_1, 0, x_3)$ will not change?

(b) Solve the problem after adding the new constraint: $-2x_1 + 3x_2 + 2x_3 \le 22$

Department of Mathematics
Comprehensive Examination
2024 Fall Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G be a group.
   (a) If H_i is a subgroup for all $i \in I$, prove $\cap_{i\in I} H_i$ is a subgroup of G.
   (b) If H_i is a **normal subgroup** for all $i \in I$, prove that $\cap_{i\in I} H_i$ is a normal subgroup of G.
   (c) Provide a counterexample to show that if H_i are subgroups for all $i \in I$, then $\cup_{i\in I} H_i$ is not necessarily a subgroup of G. Write a justification to explain why your counterexample shows the claim is not true.
2. Let R and S be commutative rings with "unity" (by which we mean a multiplicative identity) and let $\varphi : R \to S$ be a ring homomorphism.
   (a) Let J be a subring of S. Prove the inverse image of J, $\varphi^{-1}(J) = \{r \in R : \varphi(r) \in J\}$, is a subring of R.
   (b) Let J be a left ideal of S. Prove the inverse image of J, $\varphi^{-1}(J)$, is a left ideal of R.
   (c) Let P be a prime ideal of S. Prove that *either* $\varphi^{-1}(P)$ is a prime ideal of R *or* $\varphi^{-1}(P) = R$.
3. Consider $\mathbb{Z}$ as a group under addition.
   (a) Consider the group $\mathbb{Z}_3$ under addition modulo 3. Prove the quotient group $\mathbb{Z}/3\mathbb{Z}$ and the group $\mathbb{Z}_3$ are isomorphic as groups.
   (b) A subgroup H of a group G is said to be a *maximal subgroup* if there does not exist a proper subgroup N of G such that $H \leq N \leq G$. Prove $3\mathbb{Z}$ is a maximal subgroup of $\mathbb{Z}$.

Math 630

4. Let $\{f_n\}$ be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Prove that $\{f_n\}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $m, n \geq N$ and for all $x \in A$ we have $|f_n(x) - f_m(x)| < \epsilon$.
5. Compute the Fourier series for the function below. Fully justify your calculations.

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, 0] \\ 3 & \text{if } x \in (0, \pi] \end{cases}$$

6. Let $C([0,1])$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$ and give $C([0,1])$ the norm

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\}.$$

Consider the map $T : C([0,1]) \to \mathbb{R}$ given by

$$T(f) = \sum_{k=1}^{n} c_k f(t_k)$$

where $n \in \mathbb{N}$ is fixed, $c_k \in \mathbb{R}$ are fixed, and $t_k \in [0,1]$ are fixed. Prove that T is a bounded operator. (You may assume that it is linear).

Math 670

7. Let $g(x) = 4 - 2e^{-x}$

   (a) Prove that the equation $2x = g(x)$ has exactly two real solutions.

   (b) Choose one of the two solutions. Use Newton's Method to find an approximation with an absolute error of less than 10^{-6}.

8. (a) Explain how to solve the equation $A\mathbf{x} = \mathbf{b}$ using an LU decomposition.

   (b) Write the matrix form you would use to numerically solve the following partial differential equation using backward-difference in time and central-difference in space.

$$\frac{\partial u}{\partial t} - \frac{1}{16}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, 0 < t;$$
$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = 2\sin(2\pi x)$$

9. Show that

   (a) The Jacobi iteration converges for

$$A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

   but the Gauss-Seidel iteration does not converge.

   (b) The Gauss-Seidel iteration converges for

$$A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

   but the Jacobi iteration does not converge.