Department of Mathematics
Comprehensive Examination
2025 Fall Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes. You must answer at least one question from each class.

Time: 2 hours

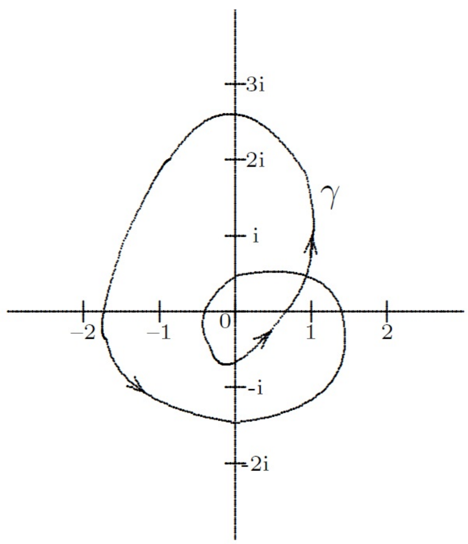
Math 640: Complex Variables

1. Let D be an open connected subset of $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be analytic in D. Prove that if $\overline{f}$ is analytic in D, then f is constant in D.

2. Evaluate

$$\int_\gamma \frac{(z-1)dz}{z(z^2+9)(z+1)^2}$$

where γ is the contour illustrated below



3. Find the image under the transformation

$$w = \frac{z+1}{z-1}$$

of

(a) The imaginary axis.

(b) The unit circle.

Department of Mathematics
Comprehensive Examination
2025 Fall Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes. You must answer at least one question from each class.

Time: 2 hours

Math 660: Topology

4. **(Full proof)** Let $X = \{x_1, x_2, x_3\}$ be a set with three distinct elements. Consider the topological space $(X, \mathcal{T}_{indiscrete})$. Let $A = \{x_1\}$ and $B = \{x_1, x_2\}$.
   (a) Prove that x_1 is an isolated point of A.
   (b) Prove that x_1, x_2 and x_3 are limit points of B.

5. Recall the following definitions discussed in class:

   Definitions: Let $(X, \mathcal{T})$ be a topological space.
   (a) The **double headed snake:** Let $\mathbb{R}_{+00}$ be the set consisting of $\mathbb{R}_+$ (the positive real numbers) together with two points which we'll call $0'$ and $0''$. Put a topology on it generated by a basis consisting of all intervals in $\mathbb{R}_+$ of the form (a, b) or else of the form $(0, b) \cup \{0'\}$ *or* $(0, b) \cup \{0''\}$ for $a, b \in \mathbb{R}_+$.
   (b) X is **Hausdorff**, or a T_2-**space**, if and only if for every pair x, y of distinct points there are disjoint open sets U, V such that $x \in U$ and $y \in V$.
   (c) A subset A of X is **compact** if and only if every open cover of A has a finite subcover.

   (a) **(Full proof)**. Prove that $\mathbb{R}_{+00}$ is not Hausdorff.
   (b) **(Full proof)**. Find a nontrivial set A in $\mathbb{R}_{+00}$ that is compact but not closed.

6. Recall the following definitions discussed in class:

   Definitions: Let X be a topological space.
   (a) X is **connected** if and only if X is not the union of two disjoint non-empty open sets.
   (b) A **path** from x to y in a space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.
   (c) A space X is **path connected** if and only if every pair of points in X can be joined by a path in X.

   (a) **(Full proof)** Prove that the continuous image of a connected space is connected.
   (b) **(Explain your reasoning)** Give an example of a connected subset of $\mathbb{R}^2_{std}$ whose closure is not path-connected.

Department of Mathematics
Comprehensive Examination
2025 Fall Semester "Choose 2" Classes

Directions: You will answer TWO questions from a total of three questions below.

Time: 2 hours

Math 680: Linear Programming

7. Prove or disprove, using the Complementary Slackness Theorem:
(1,3,4) is the optimal solution to the following linear programming problem.

$$\begin{array}{lrrrl} \text{Maximize} & 10x_1 & +12x_2 & +8x_3 & \\ \text{Subject to} & 3x_1 & +5x_2 & +x_3 & \leq 22 \\ & 2x_1 & -x_2 & +4x_3 & \leq 15 \\ & x_1 & +2x_2 & +6x_3 & \leq 32 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Solve the following minimization problem using the Simplex method. Please show all your work.

$$\begin{array}{lrrrl} \text{minimize} & 5x_1 & -3x_2 & +2x_3 & \\ \text{subject to} & 2x_1 & +5x_2 & +x_3 & \leq 30 \\ & 3x_1 & -x_2 & -4x_3 & \geq 15 \\ & 4x_1 & +2x_2 & +x_3 & \geq 18 \\ & & & x_j & \leq 0 \end{array}$$

9. Consider the following problem.

$$\begin{array}{lrrrl} \text{minimize} & -6x_1 & -3x_2 & -9x_3 & \\ \text{subject to} & -3x_1 & -2x_2 & +8x_3 & \leq 10 \\ & 4x_1 & +10x_2 & +5x_3 & \leq 12 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

The final tableaux is

	x_1	x_2	x_3	x_4	x_5	
x_3	0	$\frac{22}{47}$	1	$\frac{4}{47}$	$\frac{3}{47}$	$\frac{76}{47}$
x_1	1	$\frac{90}{47}$	0	$-\frac{5}{47}$	$\frac{8}{47}$	$\frac{46}{47}$
	0	$\frac{597}{47}$	0	$\frac{6}{47}$	$\frac{75}{47}$	$\frac{960}{47}$

Solve the following using sensitivity analysis.

(a) Find the *range* that the value of $b_1 = 10$ can have without changing the optimal solution. That is, x_1 and x_3 will be basic variables, although their final values will be altered.

(b) Find the *range* that the coefficient of x_1 in the objective function can have without changing the optimal solution. That is, x_1 and x_3 will retain their optimal values, but the value of z will be altered.

(c) Add in the following constraint and give the new solution.

$$2x_1 + 2x_2 + 4x_3 \leq 8$$

Department of Mathematics
Comprehensive Examination
2025 Fall Semester Core Classes

Time: 2.5 hours
Directions:

- You must answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Suppose we are trying to predict the weather. From past data we know that if it is sunny today, then there is a 0.70 chance to staying sunny tomorrow, 0.20 chance of turning cloudy, 0.08 chance of becoming rainy, and 0.02 chance of becoming stormy.

   Consider the four states of weather as `1=sunny, 2=cloudy, 3=rainy, 4=stormy.` Thus, the first column of a transition matrix from state 1 to states 1-4 is given by the vector
   $$\begin{bmatrix} 0.70 \\ 0.20 \\ 0.08 \\ 0.02 \end{bmatrix}$$
   In a similar fashion we can create the full Markov matrix:
   $$P = \begin{bmatrix} 0.70 & 0.40 & 0.25 & 0.15 \\ 0.20 & 0.40 & 0.45 & 0.35 \\ 0.08 & 0.15 & 0.20 & 0.30 \\ 0.02 & 0.05 & 0.10 & 0.20 \end{bmatrix}$$

   (a) Given that it is cloudy today, what are the chances that it will be rainy two days from now?

   (b) In the long run, what percentage of the time will the weather be sunny, cloudy, rainy, and stormy? In other words, what percentage of time will the weather be in each state?

2. Suppose in Juan's world he uses coordinates in basis B where
   $$B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$
   whereas Carolyn prefers to use the standard basis for $\mathbb{R}^3$, which will be denoted using the subscript S. Let $\mathbf{v}$ be given below,
   $$[\mathbf{v}]_S = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

(a) Find $[\mathbf{v}]_B$

(b) Find the matrix $P_{S\to B}$ that converts from Carolyn's world (standard basis) to Juan's world (basis B).

(c) Confirm that $[\mathbf{v}]_B = P_{S\to B}[\mathbf{v}]_S$

(d) Suppose we have a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose standard matrix is

$$[T]_S = A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

If $[\mathbf{v}]_S$ is transformed by T, what are the transformed coordinates in Juan's world?

3. Suppose a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the following properties:

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

(a) Determine the matrix A associated with the linear transformation T.

(b) Determine the stretch factors and stretch directions of the transformation T.

Math 630

4. Let $h(x) = \displaystyle\sum_{n=1}^{\infty} \frac{3}{x^2 + 2n^2}$. Prove that h is a continuous function on $\mathbb{R}$.

5. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed vector spaces. Let $T : X \to Y$ be a linear map.

(a) What does it mean for T to be bounded? (State the definition of a bounded linear operator).

(b) Prove that if T is continuous at $x = 0$, then T is bounded.

6. Consider the set of continuous functions on a closed interval

$$C([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous}\}$$

with norm given by

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}.$$

Prove that $C([a, b])$ with $\|\cdot\|_\infty$ is complete. In other words, prove that if $\{f_n\}$ is a Cauchy sequence of functions in $C([a, b])$, then they converge in the $\|\cdot\|_\infty$ norm to a function $f \in C([a, b])$.

(You may assume that $C([a, b])$ is a vector space.)

7. (a) State the conditions on the function g such that the fixed-point sequence defined by $p_{n+1} = g(p_n)$ converges to a unique fixed point p.

   (b) Prove that the nonlinear equation
$$f(x) = x^3 - 2x + 1 = 0$$
   has three roots $p_1 < p_2 < p_3$.

   (c) Two equivalent fixed point problems for f are:
$$g_1(x) = \frac{1}{2}(x^3 + 1) \text{ and } g_2(x) = \sqrt{2 - \frac{1}{x}}.$$
   In the graphs on the last page, each of these functions is plotted near the second largest root of f ($p_2 \approx .6$) along with the line $y = x$.

   For each graph, draw the fixed point iteration scheme (i.e. sketch the cobweb/boxes) corresponding to your choice of initial guess p_0. Then, explain using the conditions from part (a), why the fixed point iteration scheme converges or diverges.

8. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & -4 \\ 2 & -4 & 9 \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ 4 \\ -9 \end{pmatrix}$.

   (a) Factor the matrix A using your choice of factorization.

   (b) Using the factorization obtained in (a) solve the system $Ax = b$.

9. Prove that if A is strictly row diagonally dominant, then the Jacobi method converges for any choice of the initial approximation $x^{(0)}$.

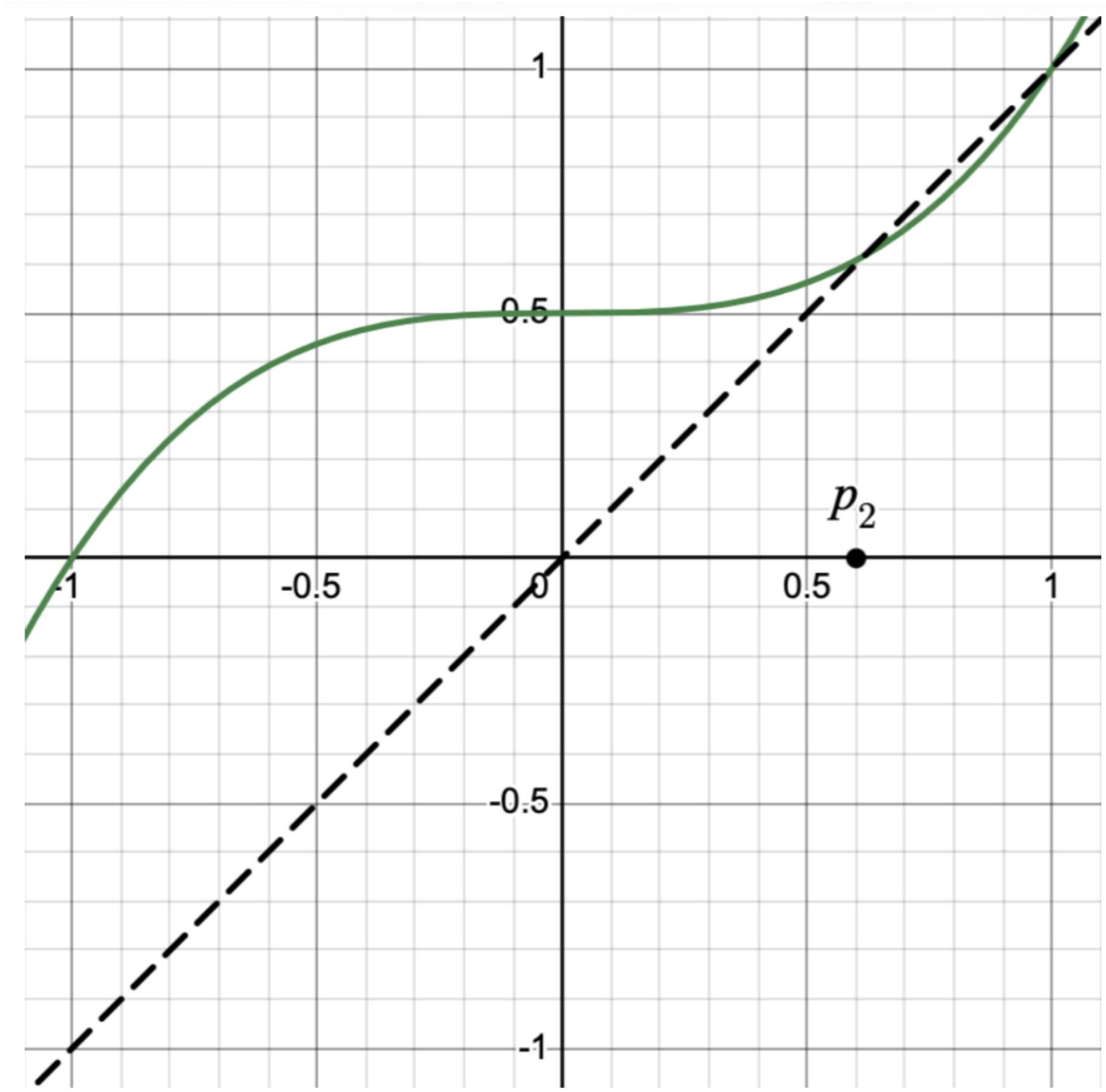


Figure 1: $g_1(x) = \frac{1}{2}(x^3 + 1)$

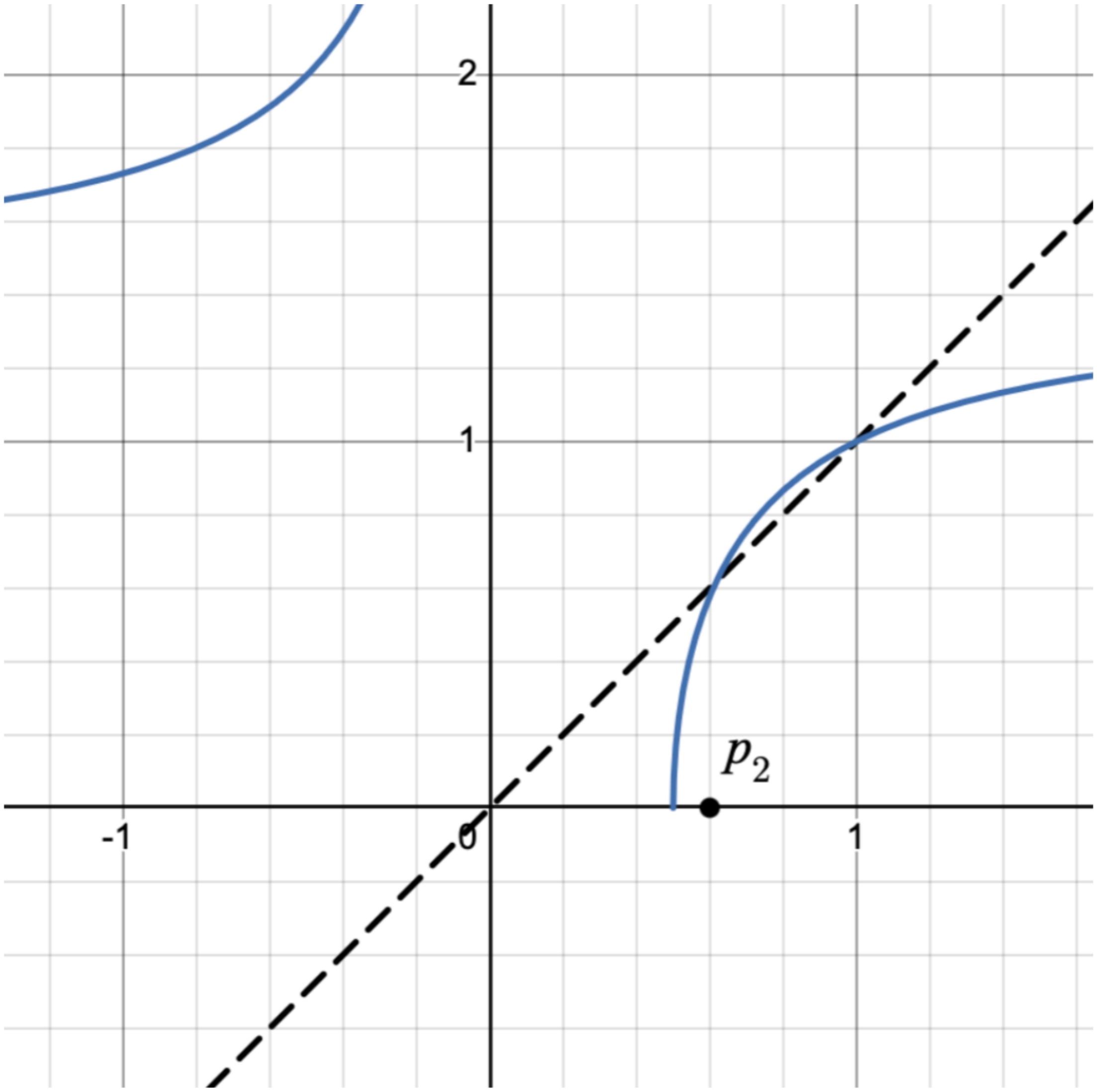


Figure 2: $g_2(x) = \sqrt{2 - \frac{1}{x}}$