Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2015 Spring

Complex Analysis

1. Find a linear fractional transformation which maps the half plane $\{a + bi \mid a > b\}$ onto the disk $\{a + bi \mid a^2 + b^2 < 4\}$, and determine its inverse.

2. Find all solutions $x + iy$ of the equation $\sin z = 2i$.

3. Let f be a function analytic in an open set $A \subset \mathbf{C}$. Prove that if Re f is constant in A, then f is constant in A.

4. Let f be an entire function and $a, b \in \mathbf{C}$ with $|a| < R$ and $|b| < R$.

   a. Prove:
   $$f(a) - f(b) = \frac{a-b}{2\pi i} \int_{|z|=R} \frac{f(z)\,dz}{(z-a)(z-b)}$$

   b. Use part a. to prove Liouville's theorem: each bounded entire function is constant.

Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2015 Spring

Real Analysis

1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that $\lim_{n\to\infty} a_n = L \neq 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

   (You must use an "ε–N" argument.)

2. Prove that each Cauchy sequence in a metric space is bounded.

3. Let $f : \mathbf{R}^2 \to \mathbf{R}$ as follows.

$$f(x,y) = \begin{cases} x, & y = 0 \\ y, & x = 0 \\ 1, & \text{elsewhere} \end{cases}$$

   (a) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

   (b) Prove that f is not differentiable at $(0,0)$ by showing that f is not continuous at $(0,0)$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ as follows.

$$f(x) = \begin{cases} 4, & x \text{ is rational} \\ -2, & x \text{ is irrational} \end{cases}$$

   Prove or disprove that f is Riemann integrable on any interval $[a,b] \subset \mathbf{R}$.

Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2015 Spring

Topology

1. Let $\mathcal{B}_l = \{[a, b) \mid a, b \in \mathbf{R} \text{ and } a < b\}$ be a collection of subsets of $\mathbf{R}$.
   (a) Prove that $\mathcal{B}_l$ is a basis for a topology $\mathbf{R}_l$ of $\mathbf{R}$ that is strictly finer than the standard topology on $\mathbf{R}$.
   (b) Determine whether the topological space $\mathbf{R}_l$ is connected. Justify your answer.

2. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps.
   Prove: If Y is Hausdorff, then $A = \{x \in X \mid f(x) = g(x)\}$ is closed in X.

3. (a) Let C and D be disjoint nonempty open subsets of a topological space X such that $X = C \cup D$. Prove: If Y is a connected subspace of X, then $Y \subset C$ or $Y \subset D$.
   (b) Let $(A_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset \ \forall n \geq 1$. Use (a) to prove that $\bigcup\{A_n \mid 1 \leq n\}$ is connected.

4. Prove the "tube lemma": Let X and Y be topological spaces and let x be a point in X. If Y is compact and N is an open neighborhood of $\{x\} \times Y$ in $X \times Y$, then there exists an open neighborhood U of x in X such that $U \times Y \subset N$.

Department of Mathematics and Computer Science

Comprehensive Examination–Option III

2015 Spring

Applied Analysis

1. Find the general solution of the following initial value problem.

$$X' = \begin{bmatrix} 5 & -1 \\ 9 & -1 \end{bmatrix} X, \qquad X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. Use power series to find the solution about the point $x_0 = 0$ of the following equation. Give at least three nonzero terms of the series solution. Also, determine the radius of convergence.

$$(x^2 - 1)y'' - y = 0, \quad y(0) = 0,\ y'(0) = 1$$

3. Prove that each Cauchy sequence in a metric space is bounded.

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ as follows.

$$f(x, y) = \begin{cases} x, & y = 0 \\ y, & x = 0 \\ 1, & \text{elsewhere} \end{cases}$$

   (a) Find $\dfrac{\partial f}{\partial x}(0, 0)$ and $\dfrac{\partial f}{\partial y}(0, 0)$.

   (b) Prove that f is not differentiable at $(0, 0)$ by showing that f is not continuous at $(0, 0)$.

Department of Mathematics and Computer Science

Comprehensive Examination–Option III

2015 Spring

Numerical Analysis

1. (a) Prove that there exists exactly one solution in the interval $[0, \infty)$ of the equation $xe^x = 3$.

   (b) Find an approximation β of the exact solution α such that $|\alpha - \beta| < 10^{-6}$.

   (c) Prove that your approximation β is in fact within 10^{-6} of (the exact) α.

   Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. (a) Determine a value of n such that using the composite Simpson's rule with n subintervals of $[1, 3]$ will give an approximation of

$$\int_1^3 (\ln x + x^3 - 2x + 1)\, dx$$

   with an error not exceeding 10^{-3}. Recall that the error term for $\int_a^b f$ is $\frac{b-a}{180} h^4 |f^{(4)}(\xi)|$ where $h = (b-a)/n$ and $\xi \in (a, b)$.

   (b) Using your value of n find an approximation of the integral above. The composite Simpson's rule can be written as follows.

$$\int_a^b f \approx \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]$$

   Here $h = (b-a)/n$, and $x_i = a + ih$ for $i = 0, 1, 2, \ldots, n$.

3. Let I be the $n \times n$ identity matrix.

   (a) Prove that $\|I\| \geq 1$ for each matrix norm.

   (b) Use the result of part (a) to prove that if A is a nonsingular $n \times n$ matrix, then $\mathrm{cond}(A) \geq 1$.

4. Write an algorithm to perform a single step of Gauss-Seidel iteration on an $n \times n$ pentadiagonal system $(n \geq 3)$ of the following form.

$$\begin{pmatrix}
a_1 & d_2 & e_3 & 0 & . & . & . & . & . & 0 \\
b_1 & a_2 & d_3 & e_4 & . & & & & & . \\
c_1 & b_2 & a_3 & d_4 & e_5 & . & & & & . \\
0 & c_2 & b_3 & a_4 & d_5 & e_6 & . & & & . \\
. & \ddots & \ddots & \ddots & . & . & . & . & . & . \\
. & & \ddots & \ddots & \ddots & . & . & . & . & . \\
. & & & . & . & . & . & . & . & 0 \\
. & & & . & . & . & . & . & . & e_n \\
. & & & & & 0 & c_{n-3} & b_{n-2} & a_{n-1} & d_n \\
. & & & & & & 0 & c_{n-2} & b_{n-1} & a_n
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \\ \vdots \\ . \\ x_{n-1} \\ x_n \end{pmatrix}
=
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ \vdots \\ \vdots \\ . \\ y_{n-1} \\ y_n \end{pmatrix}$$

Department of Mathematics and Computer Science

Comprehensive Examination–Option III

2015 Spring

Linear Programming

1. Use the Primal Simplex method to solve the following minimization problem.

$$\begin{aligned}
\text{minimize} \quad & -4x_1 + 5x_2 - 6x_3 + 3 \\
\text{subject to} \quad & 3x_1 + 4x_2 + 6x_3 \leq 8 \\
& -x_1 + 3x_2 + 2x_3 \geq 4 \\
& 2x_1 + 2x_2 + x_3 \geq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

2. Consider the following minimization problem.

$$\begin{aligned}
\text{minimize} \quad & 9x_1 + 8x_2 + 8x_3 \\
\text{subject to} \quad & 5x_1 + 3x_2 + 3x_3 \leq 9 \\
& x_1 + 2x_2 + x_3 \leq 4 \\
& 6x_1 + 4x_3 \geq 11 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

The beginning and final tableaux in the Simplex method are given in the following table.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	5	3	3	1	0	0	9
x_5	1	2	1	0	1	0	4
a_1	6	0	4	0	0	-1	11
	9	8	8	0	0	0	0
x_1	1	6	0	2	0	$3/2$	$3/2$
x_5	0	5	0	1	1	1	2
x_3	0	-9	1	-3	0	$-5/2$	$1/2$
	0	26	0	6	0	$13/2$	$-35/2$

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer the questions.

(a) What is the range on the coefficient of x_1 such that the basis variables do not change?

(b) What is the range on the coefficient of x_2 such that the basis variables do not change?

(c) What is the range on the right hand side, $b_3 = 11$, of the third constraint such that the basis variables do not change?

Linear Programming–continued

3. Solve the following transportation problem and give the final cost. The supplies are listed along the left, and the demands are listed along the top.

	50	75	100	125
55	29	10	22	11
100	31	18	32	16
55	32	35	37	33
65	28	12	20	17
75	30	19	30	15

4. Consider the following maximization problem.

$$\begin{array}{rrl} \text{maximize} & 4x_1 + 4x_2 + 3x_3 & \\ \text{subject to} & 2x_1 - 2x_2 + x_3 & \leq 10 \\ & 6x_1 + 3x_2 - x_3 & \leq 16 \\ & 3/2\,x_1 + 4x_2 + 3x_3 & \leq 15 \\ & -2x_1 + 6x_2 + 2x_3 & \leq 10 \\ & x_j & \geq 0 \end{array}$$

Using the Complementary Slackness Theorem prove or disprove the following statement.

$$\left[\frac{42}{13}, 0, \frac{44}{13}\right] \text{ is an optimal solution.}$$

Department of Mathematics and Computer Science

Comprehensive Examination–Option III

2015 Spring

Probability

1. Suppose we have a game where, at each turn, a person takes either one step forward or one step back. X_i measures how many steps the person is ahead of the origin (that is, negative values indicate the person is behind the origin) at the ith step. At the first trial $X_0 = 0$. The probability of moving forward (that is, $X_i = X_{i-1} + 1$), p_f are as follows.

$$p_f = \begin{cases} 0.25 & \text{if } X_i > 0 \\ 0.5 & \text{if } X_i = 0 \\ 0.6 & \text{if } X_i < 0 \end{cases}$$

   (a) Give the probability distribution for X_2.
   (b) Give the probability distribution for X_3.
   (c) What are the expected value and variance of X_3?
   (d) Give the probability distribution for X_3 if $X_1 = 1$.
   (e) What are the expected value and variance of X_3 if $X_1 = 1$?

2. Nicol (1994) defined the following probability density function for continuous X.

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ c/x^3 & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c such that $f(x)$ is a probability density function.
   (b) Find $E(X)$.
   (c) Find the median of X.
   (d) Find $P(0.5 \leq X \leq 1.5)$.

3. A total of four buses carrying 145 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 30, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the selected student. One of the four bus drivers is also randomly selected. Let Y denote the number of students on his/her bus.
   (a) Without calculating the numbers, which of the expected values, $E(X)$ or $E(Y)$, do think is larger? Why?
   (b) Compute $E(X)$ and $E(Y)$.
   (c) Compute the variance of X and the variance of Y, $Var(X)$ and $Var(Y)$.

Probability–continued

4. Company A has just developed a diagnostic test for a certain disease. The disease afflicts 1% of the population. The *sensitivity* of the test is the probability of someone testing positive, given that he or she has the disease, $P(+|D)$, and the *specificity* of the test is the probability that someone tests negative, given that he or she does not have the disease, $P(-|D^c)$. Assume that the sensitivity and specifity are each 95%.

   Company B, which is a rival of Company A, offers a competing test for the disease. Company B claims that their test is faster and less expensive to perform than the test from Company A, is less painful, and yet has a higher overall success rate, where the overall success rate is defined as the probability that a randomly selected person is diagnosed correctly.

   (a) The test from Company B can be described and performed very simply: no matter who the patient is, diagnose that he or she does not have the disease. Check whether the claim of Company B about overall success rates is true.

      i. Compute $P(D|+)$ and $P(D^c|-)$ for Company A.

      ii. Compute $P(D|+)$ and $P(D^c|-)$ for Company B.

      iii. Compare.

   (b) Explain why the test from Company A may still be useful.

   (c) Company A wants to develop a new test such that the overall success rate is higher than that of Company B.

      i. If the sensitivity and specificity are equal, how high does the sensitivity have to be to reach this goal?

      ii. If they can get the sensitivity equal to 1, how high does the specificity have to be to achieve the goal?

      iii. If they can get the specificity equal to 1, how high does the sensitivity have to be to achieve the goal?