Department of Mathematics

Comprehensive Examination–Option I

2016 Spring

Algebra

1. Let $G = \{\tau_{ab} : \mathbf{R} \to \mathbf{R} \mid a, b \in \mathbf{R} \text{ and } a \neq 0\}$ be the group under the usual function composition, where $\tau_{ab}(x) = ax + b,\ \forall x \in \mathbf{R}$. Let $\mathbf{R}^*$ be the group of all nonzero real numbers under the usual real number multiplication. Define $\phi : G \to \mathbf{R}^*$ by $\phi(\tau_{ab}) = a,\ \forall \tau_{ab} \in G$.
   (a) Prove that ϕ is a group homomorphism.
   (b) Determine the kernel and image of ϕ.
   (c) Use part (b) to prove that $N = \{\tau_{1b} \mid \tau_{1b} \in G\}$ is a normal subgroup of G, and G/N is abelian.

2. Let R be a commutative ring, and let $N = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}$. Prove that N is an ideal of R.

3. Let F be a field of characteristic $p \neq 0$. Prove that $(x + y)^p = x^p + y^p$ for each $x, y \in F$.

4. Let A be a real $n \times n$ skew-symmetric matrix (*i.e.*, $A^T = -A$). Prove that each eigenvalue of A is a real multiple of $i = \sqrt{-1}$.

Department of Mathematics

Comprehensive Examination–Option I

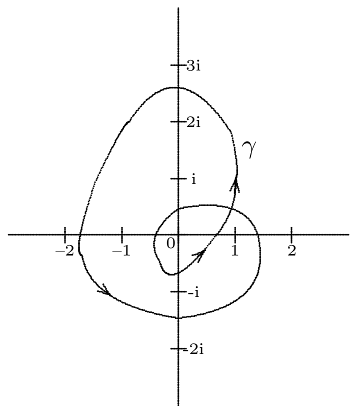
2016 Spring

Complex Analysis

1. Evaluate

$$\int_\gamma \frac{(z-1)\,dz}{z(z^2+9)(z+1)^3}$$

where γ is the contour illustrated.



2. Find all Laurent expansions of $f(z) = \dfrac{4}{z^2+2z-3}$ centered at $z_0 = 1$.

3. Find the linear fractional transformation $w = T(z) = \dfrac{az+b}{cz+d}$ which maps $z_1 = -1,\ z_2 = 0$, and $z_3 = 1$ to $w_1 = 2i,\ w_2 = 2$, and $w_3 = -2$, respectively.

4. Let f be an entire function such that $f\left(\dfrac{i}{n}\right) = \dfrac{1}{n^3}$ for each $n = 1,\ 2,\ 3,\ \ldots$. Find $f(2+i)$ and explain why your answer is correct.

Department of Mathematics

Comprehensive Examination–Option I

2016 Spring

Real Analysis

1. (a) Complete the $\varepsilon - N$ definition:

   A sequence (x_n) of real numbers <u>converges to a limit L</u> if . . . .

   (b) Use the definition to prove: if $(x_n) \to a$ where $x_n \geq 0\ \forall n$, then $(\sqrt{x_n}) \to \sqrt{a}$.

2. Let $f : [2, 4] \to \mathbf{R}$ by $f(x) = \sum_{n=1}^{\infty} \frac{x^2 \cos(n\pi x)}{n^2}$, and let $M = f([2, 4])$.

   (a) Determine whether M is compact, and justify your answer.

   (b) Determine whether M is connected, and justify your answer.

Remark. No student took the Real Analysis exam, so the committee selected no other problem for it.

Department of Mathematics

Comprehensive Examination–Option I

2016 Spring

Topology

1. Let A be a subset of a topological space X. The *boundary* of A, $\partial A = \overline{A} \cap \overline{A^c}$, is the intersection of the closure of A and the closure of its complement.
   (a) Prove that ∂A and A° are disjoint, where A° is the interior of A.
   (b) Prove that $\overline{A} = \partial A \cup A^\circ$.

2. Let X be an infinite topological space with co-finite topology; *i.e.*, $A \subset X$ is open iff $A = \emptyset$ or $X - A$ is a finite set.
   (a) Prove that X is not Hausdorff.
   (b) Prove that X is connected.

3. Prove that the continuous image of a compact topological space is compact, and then prove that if X and Y are topological spaces such that $X \times Y$ is compact, then each of X and Y is compact.

4. If A is a subset of a topological space X, then A is a retract of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Prove that if X is Hausdorff and A is a retract of X, then A is a closed subset of X.

Department of Mathematics

Comprehensive Examination–Option III

2016 Spring

Applied Analysis

1. Find the first three nonzero terms of two linearly independent power series solutions based at the origin for the differential equation $4xy'' + 2y' - y = 0$.

2. The 1-dimensional heat equation for temperature function $u(x,t)$ is given by $u_t = k\, u_{xx}$ where k is a positive constant. Use the method of separation of variables to reduce this equation to a pair of ordinary differential equations. Justify each step in your reduction.

3. (a) Complete the $\varepsilon - N$ definition:

   A sequence (x_n) of real numbers <u>converges to a limit L</u> if . . . .

   (b) Use the definition to prove: if $(x_n) \to a$ where $x_n \geq 0\ \forall n$, then $(\sqrt{x_n}) \to \sqrt{a}$.

4. Let $f : [2,4] \to \mathbf{R}$ by $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^2 \cos(n\pi x)}{n^2}$, and let $M = f([2,4])$.

   (a) Determine whether M is compact, and justify your answer.

   (b) Determine whether M is connected, and justify your answer.

Department of Mathematics

Comprehensive Examination–Option III

2016 Spring

Numerical Analysis

1. (a) Prove that there exist exactly two real solutions, one negative and one positive, of the equation $e^x = 3x + 2$.
   (b) Find an approximation β of the positive solution α such that $|\alpha - \beta| < 10^{-6}$.
   (c) Prove that your approximation is in fact within 10^{-6} of (the exact) α.
   Note: For this problem you may not use any graphing or root finding capabilities of your calculator.

2. Suppose $f : [0, 1] \to \mathbf{R}$ by $f(x) = e^{-x}$.
   (a) Find the polynomial $p(x)$ such that degree$(p(x)) \leq 2$, $p(0) = f(0)$, $p(1/2) = f(1/2)$, and $p(1) = f(1)$.
   (b) Using the formula from the interpolating polynomial error theorem show that
      i. $e^{-1/3} - p(1/3) < 0$, and
      ii. $|e^{-1/3} - p(1/3)| < 1/162 < 0.00625$.

3. Let
$$A = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}.$$
   It can be shown that A is positive definite.
   (a) Construct T_j, the associated iteration matrix for Jacobi iteration.
   (b) Show that -1 is an eigenvalue of T_j.
   (Note: You may not use any eigenvalue capabilities of your calculator.)

4. Suppose $n \geq 3$ and A is an $n \times n$ matrix of the following form.
$$\begin{pmatrix} a_1 & 0 & d_3 & & & & & & \\ b_1 & a_2 & 0 & d_4 & & & & & \\ & b_2 & a_3 & 0 & d_5 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & b_{n-3} & a_{n-2} & 0 & d_n \\ & & & & & & b_{n-2} & a_{n-1} & 0 \\ & & & & & & & b_{n-1} & a_n \end{pmatrix}$$
   Write an efficient algorithm which constructs a lower triangular L and upper triangular U such that
   i. $A = LU$, and
   ii. L and U are constructed with significant diagonal vectors as follows.
$$L = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ f_1 & 1 & 0 & & & & \cdot \\ 0 & f_2 & 1 & 0 & & & \cdot \\ 0 & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & f_{n-1} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} h_1 & p_2 & q_3 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & h_2 & p_3 & q_4 & 0 & & & \cdot \\ \cdot & 0 & h_3 & p_4 & q_5 & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & \cdot & \cdot & \cdot & q_n \\ \cdot & & & & & 0 & h_{n-1} & p_n \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & h_n \end{pmatrix}$$
   You should assume that pivoting is unnecessary. Exploit the sparsity pattern by writing everything in terms of the given vectors. (No doubly subscripted variable should appear in your solution.)

Department of Mathematics

Comprehensive Examination–Option III

2016 Spring

Linear Programming

1. Use the Primal Simplex method to solve the following minimization problem.

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 4x_2 + 5x_3 + 9 \\ \text{subject to} \quad & x_1 + 2x_2 - 3x_3 \geq 8 \\ & -2x_1 + 4x_2 + 4x_3 \leq 15 \\ & x_1 + x_2 + 2x_3 \geq 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2. Consider the following maximization problem.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 \leq 9 \\ & 4x_1 + x_2 - x_3 \leq 12 \\ & 3x_1 + 3x_2 + 2x_3 \leq 24 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The beginning and final tableaux in the Simplex method are given in the following table.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	9
x_5	4	1	-1	0	1	0	12
x_6	3	3	2	0	0	1	24
	-3	-2	1	0	0	0	0
x_4	0	0	$4/3$	1	0	$-1/3$	1
x_1	1	0	$-5/9$	0	$1/3$	$-1/9$	$4/3$
x_2	0	1	$11/9$	0	$-1/3$	$4/9$	$20/3$
	0	0	$16/9$	0	$1/3$	$5/9$	$52/3$

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer the questions.

(a) Suppose the coefficient c_1 of x_1 can be changed by a value of 2λ and the coefficient c_2 of x_2 can be changed by a value of λ. Give the range on λ, and thus on c_1 and c_2, such that the same basic variables are in the final solution.

(b) Suppose the right hand side constants in the second and third constraints change to 15 and 9, respectively. Thus the constraints are now the following.

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 + 2x_3 \leq 9 \\ & 4x_1 + x_2 - x_3 \leq 15 \\ & 3x_1 + 3x_2 + 2x_3 \leq 9 \end{aligned}$$

Does the solution change? If so, find the new solution.

Linear Programming–continued

3. Using the Complementary Slackness Theorem prove or disprove the following statement. $[3, 0, 5]$ is the optimal solution of the maximization problem below.

$$\begin{aligned} \text{maximize} \quad & 9x_1 + 4x_2 + 8x_3 \\ \text{subject to} \quad & 3x_1 - \ \ x_2 - 6x_3 \leq 26 \\ & 6x_1 + 3x_2 + 2x_3 \leq 28 \\ & -3x_1 + 2x_2 + 4x_3 \leq 11 \\ & x_j \geq \ 0 \end{aligned}$$

4. Suppose $\mathbf{x}_0 \in \mathbf{R}^n$ is a feasible solution to the following problem.

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} = z \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Suppose $\mathbf{y}_0 \in \mathbf{R}^m$ is a feasible solution to the dual problem, as follows.

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^T\mathbf{y} = v \\ \text{subject to} \quad & A^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

Prove that $\mathbf{c}^T\mathbf{x}_0 \leq \mathbf{b}^T\mathbf{y}_0$.

Department of Mathematics

Comprehensive Examination–Option III

2016 Spring

Probability

1. MEGA Millions is a lottery game available to players who are 18 and older in certain states. In the game 5 white balls numbered from 1 to 75 are drawn from a drum without replacement. A sixth gold "MEGA Ball" (numbered 1 to 15) is drawn from a separate drum. The MEGA Ball is considered separate from the white balls.
   (a) What is the probability that someone matches all 5 white numbers and the gold MEGA Ball number?
   (b) What is the probability that someone matches all five white numbers?
   (c) Of the five white numbers drawn in the last drawing, what is the probability that none of these is drawn in this drawing?
   (d) Of the five white numbers drawn in the last drawing, what is the probability that at least one of those is drawn in this drawing?
   (e) What is the probability that a ticket has none of the 5 white numbers or the gold MEGA Ball number?

2. The length of time in hours needed to complete a task follows the probability density function defined below. Let X = time to completion.

$$f(x) = \begin{cases} cx^2 + x, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find c.
   (b) Show that $f(x)$ is a probability density function.
   (c) Find the probability that the task can be completed in less than 1/2 hour.
   (d) Find the cumulative distribution function.
   (e) Given that Sally needs at least 15 minutes to complete the task, find the probability that Sally will take over 30 minutes to complete the task.

3. If heads is a success when we flip a coin, getting a six is a success when we roll a die, and getting an ace is a success when we draw a card from an ordinary deck of 52 playing cards, find the mean and standard deviation of the total number of successes when we
   (a) flip a fair coin, roll a balanced die, and then draw a card from a well-shuffled deck;
   (b) flip a fair coin three times, roll a balanced die twice, and then draw a card from a well-shuffled deck.

Probability–continued

4. Suppose three urns numbered 1, 2, and 3 contain, respectively, one red and one blue ball, two red and three blue balls, and four red and two blue balls. Consider an experiment consisting of the selection of an urn and followed by the draw of one ball from it. Define $B_1,\ B_2,\ B_3$ to be the event urn 1, 2, or 3 is selected; define A to be the event that a red ball is selected.
   (a) Assume equal probabilities for each urn selected:
   $P(B_1) = P(B_2) = P(B_3) = 1/3$.
   Compute the probability that a red ball is selected, $P(A)$.
   (b) If the urn number is not observed but a red ball is drawn, what is the probability that it was drawn from urn 1, urn 2, or urn 3? Compute $P(B_1|A),\ P(B_2|A)$, and $P(B_3|A)$.
   (c) Verify that the conditional probabilities in part (b) are correct by computing the sum of the three probabilities. Which urn is most likely the one from which the red ball was drawn?
   (d) Now assume the probabilities for each urn are unequal:
   $P(B_1) = 1/2,\ P(B_2) = 1/3$, and $P(B_3) = 1/6$.
   Again, if the urn number is not observed but a red ball is drawn, what is the probability that it was drawn from urn 1, urn 2, or urn 3? Compute $P(B_1|A),\ P(B_2|A)$, and $P(B_3|A)$. Which urn is most likely the one from which the red ball was drawn?