Department of Mathematics
Comprehensive Examination - Option I
2018 Spring

Algebra

1. Let G and G' be groups and $\phi : G \to G'$ be a surjective group homomorphism.

   (a) Prove that the kernel K of ϕ is a subgroup of G and normal in G.

   (b) Define $\Phi : G/K \to G'$ by $\Phi(gK) = \phi(g)\ \forall gK \in G/K$.
   Prove that Φ is a well-defined group isomorphism from G/K onto G'.

2. Let R be a commutative ring with multiplicative identity 1. Prove:

   (a) $\forall a \in R$, $\langle a \rangle = \{ar : r \in R\}$ is an ideal of R containing a.

   (b) If the only ideal of R is $\{0\}$ and R, then R is a field.

3. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients in variable x.

   (a) Prove that 3 and x are irreducible polynomials in $\mathbb{Z}[x]$.

   (b) Prove or disprove that the ideal $I = \langle 3,\, x \rangle = \{3f(x) + xg(x) : f(x),\, g(x) \in \mathbb{Z}[x]\}$ is a principal ideal.

4. Let V and W be vector spaces of field F, $T : V \to W$ be an injective linear transformation, and $\{\mathbf{v}_1,\, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ be a subset of V for some postive integer k. Prove: $\{\mathbf{v}_1,\, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is linearly independent in V if and only if $\{T(\mathbf{v}_1),\, T(\mathbf{v}_2), \cdots,\, T(\mathbf{v}_k)\}$ is linearly independent in W.

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Real Analysis

1. Given that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are Riemann integrable and $f(x) \geq g(x)$ for all $x \in [a, b]$. Prove, **directly from the definition**, that $\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$.

2. Consider a sequence $\{x_n\}_1^\infty \in \mathbb{R}$. Assume we know that the three subsequences below are convergent.
$$\{x_{2n}\}_1^\infty, \quad \{x_{2n+1}\}_1^\infty, \quad \{x_{5n}\}_1^\infty$$
Can we conclude that the sequence is convergent? Why or why not?

3. Prove that
$$g(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$
is continuous on $\mathbb{R}$.

4. Given $\{x_n\}$, a sequence of real numbers such that $|x_{n+1} - x_n| < \left(\frac{3}{4}\right)^n$ for all natural numbers n, prove that $\{x_n\}$ converges in $\mathbb{R}$.

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Applied Analysis

1. Find the solution about the point $x = 1$ to the following initial value problem.

$$xy'' + 3(x-1)y' + 6y = 0, \quad y(1) = 4, \; y'(1) = 9$$

   Please show terms out to the fifth power.

2. Find the general solution of the following initial value problem.

$$X'(t) = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} X(t), \quad X(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3. Prove that

$$g(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

   is continuous on $\mathbb{R}$.

4. Given $\{x_n\}$, a sequence of real numbers such that $|x_{n+1} - x_n| < \left(\frac{3}{4}\right)^n$ for all natural numbers n, prove that $\{x_n\}$ converges in $\mathbb{R}$.

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Numerical Analysis

1. Consider the equation $x^2 - 3\ln(x) - 4 = 0$. (Note: For this problem, you may not use any graphing or root finding capabilities of your calculator.)

   (a) Prove that the equation has exactly two solutions.

   (b) Choose one of the two solutions. Use Newton's Method to find an approximation with an absolute error of less than 10^{-6}.

2. Let
$$A = \begin{bmatrix} 2 & 5 & -5 \\ 3 & 3 & 3 \\ 8 & 8 & 4 \end{bmatrix}$$

   (a) Write out the matrix T_j corresponding to the Jacobi method in the iterative method $x_{k+1} = T_j x_k + c_j$. Will the Jacobi method converge for any initial vector x_0?

   (b) Write out the matrix T_g corresponding to the Gauss-Seidel method in the iterative method $x_{k+1} = T_g x_k + c_g$. Will the Gauss-Seidel method converge for any initial vector x_0?

3. Suppose $A \in \mathbb{R}^{5\times 5}$ is a nonsingular matrix. Normally we consider Gaussian Elimination to give a factorization of $A = LU$. For this situation, give an algorithm to factor $A = UL$ where U is an upper **unit** triangular matrix ($u_{ii} = 1$), and L is a lower triangular matrix. Assume that no pivoting is required.

4. Let $f(x) = \sin(2x)\cos x$. Approximate this function with $g(x) = c_0 + c_1 x + c_2 x^2$ such that $||g(x) - f(x)||_2$ on the interval $[0, \pi]$ is as small as possible.
(Note: you may want to use the formula $\sin(A)\cos(B) = \frac{\sin(A+B)+\sin(A-B)}{2}$)

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Linear Programming

1. Solve the following minimization problem using the primal simplex method.

$$\begin{array}{llllll} \text{mimimize} & x_1 & +\,2x_2 & +\,3x_3 & & \\ \text{subject to} & 2x_1 & -\,4x_2 & +\,5x_3 & \leq 20 \\ & 3x_1 & +\,6x_2 & +\,x_3 & \geq 10 \\ & 4x_1 & -\,x_2 & -\,2x_3 & \geq 4 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

2. Consider the following problem.

$$\begin{array}{llllll} \text{maximize} & 50x_1 & +\,80x_2 & +\,90x_3 & & \\ \text{subject to} & 5x_1 & +\,20x_2 & +\,15x_3 & \leq 200 \\ & 9x_1 & +\,15x_2 & +\,24x_3 & \leq 300 \\ & 6x_1 & +\,8x_2 & +\,15x_3 & \leq 100 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

The final tableaux is

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	$\frac{3}{16}$	$\frac{3}{40}$	0	$-\frac{1}{16}$	$\frac{35}{4}$
x_5	0	0	$\frac{15}{16}$	$-\frac{9}{40}$	1	$-\frac{21}{16}$	$\frac{495}{4}$
x_1	1	0	$\frac{9}{4}$	$-\frac{1}{10}$	0	$\frac{1}{4}$	5
	0	0	$\frac{75}{2}$	1	0	$\frac{15}{2}$	950

Solve the following using sensitivity analysis.

(a) Find the *range* that the value of $b_3 = 100$ can have without changing the optimal solution. That is, same basic variables as the original problem, although their final values will be altered.

(b) Add in the following constraint to the original problem and give the new solution.

$$6x_1 + 4x_2 + x_3 \leq 60$$

3. Using the Complementary Slackness Theorem, show whether $(0, 7, 1, 0)$ is the optimal solution to

$$\begin{array}{lllllll} \text{maximize} & 8x_1 & +\,8x_2 & -\,7x_3 & +\,5x_4 & & \\ \text{subject to} & 2x_1 & +\,4x_2 & -\,3x_3 & +\,3x_4 & \leq 25 \\ & -x_1 & +\,2x_2 & -\,x_3 & +\,5x_4 & = 13 \\ & 6x_1 & +\,8x_2 & +\,3x_3 & +\,2x_4 & \leq 60 \\ & & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

4. The latest *Game of Thrones* book has just been published, and Amazon wants to make sure that all its warehouses are well stocked so that orders can be filled in a timely manner. Suppose there are four factories in North Carolina, producing, respectively, 35, 30, 35, and 40 boxes full of the books. There are four warehouses that require stocking: California needs 30 boxes, Washington State needs 40 boxes, Arizona needs 50 boxes, and Oregon needs 20 boxes. The following chart shows the costs to ship from the respective factories in North Carolina to the various warehouses.

	CA	WA	AZ	OR
	30	40	50	20
Factory 1 = 35	15	30	14	11
Factory 2 = 30	8	6	10	7
Factory 3 = 35	13	18	16	16
Factory 4 = 40	14	17	20	15

Find the cheapest way to ship all the boxes from all the factories to all the warehouses, using a transportation algorithm.

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Probability

1. The probability of a telemarketing sales representative making a sale on a customer call is 0.10. Find the probability that

   (a) No sales are made in 10 calls.

   (b) More than 2 sales are made in 20 calls.

   The sales representatives are required to achieve a mean of at least 5 sales each day.

   (c) Find the least number of calls each day a representative should make to achieve this requirement.

   (d) Calculate the least number of calls that need to be made by a representative for the probability of at least 1 sale to exceed 0.95.

2. Let the random variable X be distributed as *Uniform(0,1)*.

$$\begin{aligned} f_X(x) &= 1 \quad 0 < x < 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

   Let $Y = -2\ln(X)$.

   (a) Determine the distribution function of Y.

   (b) Determine the density function of Y.

   (c) Determine the moment generation function of Y.

   (d) If $Y_1, Y_2, \cdots, Y_n$ are independently distributed as Y, and $Z = Y_1 + Y_2 + \cdots + Y_n$, determine the moment generating function (MGF) of the random variable Z.

3. Suppose that X is the number of heads in four flips of a coin. Let Y be the random variable $X - 2$, the difference between X and its expected value.

   (a) Compute $E(Y)$. Does it effectively measure how much we expect to see X deviate from its expected value?

   (b) Compute variance of X, $V(X)$.

   (c) What is the sum of the variances for four independent trials of one flip of a coin? (hint: first find the variance for the number of heads in one flip of a coin).

   (d) If we want to be 95% sure that the number of heads in n flips of a coin is within $\pm 1\%$ of the expected value, how big does n have to be?

4. When coded messages are sent there are sometimes errors in the transmission. In particular, Morse code uses “dots” and “dashes,” which are known to occur in the proportion of 3:4. This means that for any given symbol

$$P(\text{dot sent}) = 3/7 \qquad P(\text{dash sent}) = 4/7$$

Suppose there is some interference on the transmission line, and with probability $\frac{1}{8}$ a sent dot is mistakenly received as a dash, and a sent dash is mistakenly received as a dot with the same probability.

(a) Specify the probability P(dash received|dot sent) and P(dot received|dash sent).

(b) Compute the probability P(dot received).

(c) Compute the probability P(dot sent|dot received).

(d) Compute the probability P(dash sent|dot received).