Department of Mathematics
Comprehensive Examination
2020 Spring Semester Part 1: Core Classes

Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Time: 2.5 hours

Math 620

1. Let G and G' be groups and $\varphi : G \to G'$ be a group homomorphism with kernel K. Prove: If N' is a normal subgroup of G', then $N = \varphi^{-1}(N') = \{x \in G \mid \varphi(x) \in N'\}$ is a subgroup of G containing K and N is normal in G.

2. Let R be a commutative ring.

   (a) For a fixed nonzero element a of R, prove $S = \{x \in R : ax = 0\}$ is an ideal of R. Furthermore, a is a zero divisor if and only if $S \neq \{0\}$.

   (b) For $R = \mathbb{Z} \times \mathbb{Z}$, determine whether R is an integral domain. If not, find all its zero divisors.

Math 630

3. Let $\langle M, \rho \rangle$ denote a metric space.

   (a) State the definition of a Cauchy sequence in $\langle M, \rho \rangle$.

   (b) Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\langle M, \rho \rangle$. Prove that there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that
   $$\sum_{k=1}^{\infty} \rho(a_{n_k}, a_{n_{k+1}}) < \infty.$$

4. Consider the sequence of functions $f_n : [-\pi, \pi] \to \mathbb{R}$ given by
   $$f_n(x) = \frac{\sin(nx)}{n}, \qquad n = 1, 2, 3, \ldots$$

   (a) Find the limit function $f(x) = \lim_{n \to \infty} f_n(x)$.

   (b) Prove that the sequence f_n converges uniformly to f as $n \to \infty$.

   (c) Let $f_n'(x) = \frac{d}{dx}(f_n(x))$ denote the sequence of derivatives. Does f_n' converge uniformly to f' as $n \to \infty$? Prove your assertion.

5. Consider the equation $x^2 = 1 + \sin x$.
   (a) Prove that the equation has exactly two solutions.
   (b) Choose one of the two solutions. Use Newton's Method to find an approximation with an absolute error of less than 10^{-6}. Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Use Taylor's Theorem to prove that the 2nd derivative approximation formula is an $\mathcal{O}(h^2)$ approximation:

$$f''(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}$$

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2020 Spring Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Analysis

1. Determine all values a, b, and c for which $u(x, y) = ax^4 + bx^2y + cx^2y^2 + y^3 + y^4$ is the real part of an analytic function.

2.
   (a) For each of the functions

$$f(z) = \frac{1}{\sin z} \qquad g(z) = \sin\left(\frac{1}{z}\right) \qquad h(z) = \frac{\sin z}{z}$$

   Determine whether $z = 0$ is a removable singularity, a pole, or an essential singularity.

   (b) Suppose C is the unit circle oriented clockwise; use residues to compute

$$\oint_C \sin\left(\frac{1}{z}\right)(z^2 + 1)\,\mathrm{dz}$$

Math 660: Topology

3. Let X be a Hausdorff space. Prove each of the following.
   (a) Every one-point subset of X is closed.
   (b) If x_0 is a limit point of a subset A of X, then each open set containing x_0 contains infinitely many points of A.

4. Prove each of the following:
   (a) Every closed subspace of a compact space is compact.
   (b) Let X be a compact topological space, Y be a Hausdorff topological space, and $f : X \to Y$ be a continuous surjective map. For each subset C of Y, $f^{-1}(C)$ is closed in X implies that C is closed in Y.

Math 675: Differential Equations

5. Consider the matrix $A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

   (a) Find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

   (b) Let $\mathbf{x}_0 \in \mathbb{R}^3$. Consider the IVP

   $$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

   Find the set of all possible $\mathbf{x}_0$ giving rise to solutions with the property

   $$\lim_{t\to\infty} \mathbf{x}(t) \neq \mathbf{0}.$$

6. Consider the ODE

   $$x^2y'' + x(x + \frac{1}{2})y' + xy = 0.$$

   (a) Verify that the origin is a regular singular point.

   (b) Verify that the roots of the indicial equation do not differ by an integer.

   (c) Use the Frobenius method to find two independent solutions y_1 and y_2. Calculate the first three nonzero terms of y_1 and the first three nonzero terms of y_2 explicitly.

Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rrrrl} \max & 4x_1 & +\ 6x_2 & +\ 8x_3 & \\ \text{s/t} & 2x_1 & +\ \ x_2 & +\ 2x_3 & \geq 20 \\ & 3x_1 & +\ 3x_2 & +\ 4x_3 & \geq 10 \\ & 2x_1 & +\ 4x_2 & +\ 3x_3 & \leq 26 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Using the Complementary Slackness Theorem, show whether $(5, 0, 6, 0)$ is the optimal solution to

$$\begin{array}{lrrrrl} \text{maximize} & 27x_1 & -\ 4x_2 & +\ 18x_3 & -\ 20x_4 & \\ \text{subject to} & 5x_1 & +\ 4x_2 & -\ 2x_3 & +\ \ x_4 & \leq 16 \\ & -x_1 & -\ 3x_2 & +\ 4x_3 & +\ 2x_4 & = 19 \\ & 4x_1 & -\ 2x_2 & +\ \ x_3 & -\ 5x_4 & \leq 26 \\ & & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$