Department of Mathematics  
Comprehensive Examination  
2020 Spring Semester Part 1: Core Classes

Directions:

• You will answer FOUR questions.

• You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).

• The fourth question can come from any of the classes.

Time: 2.5 hours

Math 620

1. Let \( G \) and \( G' \) be groups and \( \varphi : G \to G' \) be a group homomorphism with kernel \( K \). Prove: If \( N' \) is a normal subgroup of \( G' \), then \( N = \varphi^{-1}(N') = \{x \in G | \varphi(x) \in N'\} \) is a subgroup of \( G \) containing \( K \) and \( N \) is normal in \( G \).

2. Let \( R \) be a commutative ring.
   (a) For a fixed nonzero element \( a \) of \( R \), prove \( S = \{x \in R : ax = 0\} \) is an ideal of \( R \). Furthermore, \( a \) is a zero divisor if and only if \( S \neq \{0\} \).
   (b) For \( R = \mathbb{Z} \times \mathbb{Z} \), determine whether \( R \) is an integral domain. If not, find all its zero divisors.

Math 630

3. Let \( \langle M, \rho \rangle \) denote a metric space.
   (a) State the definition of a Cauchy sequence in \( \langle M, \rho \rangle \).
   (b) Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence in \( \langle M, \rho \rangle \). Prove that there exists a subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \) such that
   \[
   \sum_{k=1}^{\infty} \rho(a_{n_k}, a_{n_k+1}) < \infty.
   \]

4. Consider the sequence of functions \( f_n : [-\pi, \pi] \to \mathbb{R} \) given by
   \[
   f_n(x) = \frac{\sin(nx)}{n}, \quad n = 1, 2, 3, ...
   \]
   (a) Find the limit function \( f(x) = \lim_{n \to \infty} f_n(x) \).
   (b) Prove that the sequence \( f_n \) converges uniformly to \( f \) as \( n \to \infty \).
   (c) Let \( f'_n(x) = \frac{d}{dx}(f_n(x)) \) denote the sequence of derivatives. Does \( f'_n \) converge uniformly to \( f' \) as \( n \to \infty \)? Prove your assertion.
5. Consider the equation \( x^2 = 1 + \sin x \).

   (a) Prove that the equation has exactly two solutions.

   (b) Choose one of the two solutions. Use Newton’s Method to find an approximation with an absolute error of less than \( 10^{-6} \). Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Use Taylor’s Theorem to prove that the 2nd derivative approximation formula is an \( O(h^2) \) approximation:

   \[
   f''(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2}
   \]
Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Analysis

1. Determine all values \( a, b, \) and \( c \) for which \( u(x, y) = ax^4 + bx^2y + cx^2y^2 + y^3 + y^4 \) is the real part of an analytic function.

2. (a) For each of the functions

\[
\begin{align*}
  f(z) &= \frac{1}{\sin z} \\
  g(z) &= \sin \left( \frac{1}{z} \right) \\
  h(z) &= \frac{\sin z}{z}
\end{align*}
\]

Determine whether \( z = 0 \) is a removable singularity, a pole, or an essential singularity.

(b) Suppose \( C \) is the unit circle oriented clockwise; use residues to compute

\[
\oint_C \sin \left( \frac{1}{z} \right) (z^2 + 1) \, dz
\]

Math 660: Topology

3. Let \( X \) be a Hausdorff space. Prove each of the following.

   (a) Every one-point subset of \( X \) is closed.

   (b) If \( x_0 \) is a limit point of a subset \( A \) of \( X \), then each open set containing \( x_0 \) contains infinitely many points of \( A \).

4. Prove each of the following:

   (a) Every closed subspace of a compact space is compact.

   (b) Let \( X \) be a compact topological space, \( Y \) be a Hausdorff topological space, and \( f : X \to Y \) be a continuous surjective map. For each subset \( C \) of \( Y \), \( f^{-1}(C) \) is closed in \( X \) implies that \( C \) is closed in \( Y \).
Math 675: Differential Equations

5. Consider the matrix \( A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix} \)

(a) Find the general solution to the system \( \mathbf{x}'(t) = A\mathbf{x}(t) \).
(b) Let \( \mathbf{x}_0 \in \mathbb{R}^3 \). Consider the IVP

\[
\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.
\]

Find the set of all possible \( \mathbf{x}_0 \) giving rise to solutions with the property

\[
\lim_{t \to \infty} \mathbf{x}(t) \neq \mathbf{0}.
\]

6. Consider the ODE

\[
x^2y'' + x(x + \frac{1}{2})y' + xy = 0.
\]

(a) Verify that the origin is a regular singular point.
(b) Verify that the roots of the indicial equation do not differ by an integer.
(c) Use the Frobenius method to find two independent solutions \( y_1 \) and \( y_2 \). Calculate the first three nonzero terms of \( y_1 \) and the first three nonzero terms of \( y_2 \) explicitly.

Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

\[
\text{max } 4x_1 + 6x_2 + 8x_3 \\
\text{s/t } 2x_1 + x_2 + 2x_3 \geq 20 \\
3x_1 + 3x_2 + 4x_3 \geq 10 \\
2x_1 + 4x_2 + 3x_3 \leq 26 \\
x_1, x_2, x_3 \geq 0
\]

8. Using the Complementary Slackness Theorem, show whether \((5, 0, 6, 0)\) is the optimal solution to

\[
\text{maximize } 27x_1 - 4x_2 + 18x_3 - 20x_4 \\
\text{subject to } 5x_1 + 4x_2 - 2x_3 + x_4 \leq 16 \\
-x_1 - 3x_2 + 4x_3 + 2x_4 = 19 \\
4x_1 - 2x_2 + x_3 - 5x_4 \leq 26 \\
x_1, x_2, x_3, x_4 \geq 0
\]