Department of Mathematics
Comprehensive Examination
2021 Spring Semester Part 1: Core Classes

Time: 2.5 hours **Directions:**

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G be a group and let $Z = \{x \in G \mid xg = gx \text{ for all } g \in G\}$.
   (a) Prove that Z is a normal subgroup of G (prove both that it is a subgroup **and** that it is normal).
   (b) Prove that $\phi : G \to G/Z$ defined by $\phi(g) = gZ$ is a group homomorphism.
   (c) Prove that if the quotient (factor) group, G/Z, is cyclic, then G is abelian.

2. Let R be an integral domain and let $a, b, c \in R$.
   (a) Prove that for every $c \neq 0$, if $ac = bc$ then $a = b$.
   (b) Show by giving an example that if R is a commutative ring with unity, but **not** an integral domain, than it is not true that for every $c \neq 0$, if $ac = bc$ then $a = b$.

Math 630

3. Let $< M, \rho >$ be a metric space.
   (a) Let a be a point in M and let $r \in \mathbb{R}$, $r > 0$. Prove, directly from the definition of an open set, that $O = \{x \in M \mid \rho(x, a) > r\}$ is open in M.
   (b) Let $A \subset M$ such that $< A, \rho >$ is complete. Prove that A is closed in M.

4. Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(kx)x^{2k}}{k!}$ converges to a function that is continuous on $\mathbb{R}$.

Continued $\Longrightarrow$

5. Consider the equation $e^x = 3x^2$

   (a) Prove that the equation has exactly three real solutions.

   (b) Let α be the largest of the three solutions. Use Newton's Method to find an approximation of α with an absolute error of less than 10^{-7}. Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Let A be an $n \times n$ system of the form

$$\begin{bmatrix}
a_1 & d_2 & 0 & . & . & . & . & . & . & 0 \\
b_1 & a_2 & d_3 & 0 & & & & & & . \\
c_1 & b_2 & a_3 & d_4 & 0 & & & & & . \\
0 & c_2 & b_3 & a_4 & d_5 & 0 & & & & . \\
. & . & . & . & . & . & . & & & . \\
. & & . & . & . & . & . & . & & . \\
. & & & . & . & . & . & . & . & . \\
. & & & . & . & . & . & . & . & 0 \\
. & & & & & 0 & c_{n-3} & b_{n-2} & a_{n-1} & d_n \\
0 & & & & & & 0 & c_{n-2} & b_{n-1} & a_n
\end{bmatrix}$$

with $n \geq 3$.

Write an efficient algorithm that constructs L and U where $A = LU$. This will be accomplished by modifying the b and c vectors so that L is now found in the strictly lower triangular part of A with the usual implied main diagonal of 1s. Modify the a vector so that U is now found in the upper triangular part of A.

You should assume that pivoting is unnecessary. Exploit the sparsity pattern by writing everything in terms of the given vectors.

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2021 Spring Semester Part 2: "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 660: Topology

1. Let X be a nonempty set and $(X, \mathcal{T}_{cofinite})$ be the topological space given by equipping X with the cofinite topology. Prove that $(X, \mathcal{T}_{cofinite})$ is compact.

2. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ with A nonempty. Prove that A is open in X if and only if every open subset of A (in the subspace topology) is also open in X.

   Clarification: for this question the word "subset" denotes any proper or improper subset.

Math 675: Differential Equations

3. Consider the matrix $A = \begin{pmatrix} -3 & -6 & 0 \\ 4 & 7 & 0 \\ 4 & 6 & 1 \end{pmatrix}$

   (a) Find the general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

   (b) Let $\mathbf{x}_0 \in \mathbb{R}^3$. Consider the IVP $\mathbf{x}'(t) = A\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0$ where A is given above, and $\mathbf{x}_0$ is

   $$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.$$

   Find the particular solution to this IVP.

4. Consider the ODE

   $$2x^2y'' + xy' + (x-1)y = 0.$$

   (a) Verify that the origin is a regular singular point.

   (b) Verify that the roots of the indicial equation do not differ by an integer.

   (c) Use the Frobenius method to find two independent solutions y_1 and y_2. Calculate the first four nonzero terms of y_1 and the first four nonzero terms of y_2 explicitly.

Continued $\Longrightarrow$

Math 680: Optimization

5. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rrcrcrl}
\text{minimize} & 14x_1 & + & 11x_2 & + & 12x_3 & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & \geq 18 \\
 & x_1 & + & 2x_2 & + & 4x_3 & = 30 \\
 & 6x_1 & + & 3x_2 & + & 2x_3 & \leq 36 \\
 & 4x_1 & + & 4x_2 & + & 3x_3 & \leq 36 \\
 & & & & & x_1, x_2, x_3 & \geq 0
\end{array}$$

6. Using the Complementary Slackness Theorem, show whether $(0, 0, 4, 7)$ is the optimal solution to

$$\begin{array}{rrrrrl}
\text{maximize} & 35x_1 & +\,40x_2 & +\,27x_3 & +\,21x_4 & \\
\text{subject to} & 2x_1 & +\,3x_2 & -\,4x_3 & +\,x_4 & \leq 15 \\
 & 6x_1 & -\,2x_2 & -\,3x_3 & +\,5x_4 & = 23 \\
 & -4x_1 & +\,7x_2 & +\,3x_3 & +\,6x_4 & \leq 54 \\
 & & & & x_1, x_2, x_3, x_4 & \geq 0
\end{array}$$