**Department of Mathematics**
**Comprehensive Examination**
**2022 Spring Semester "Choose 2" Classes**

**Directions:** You will answer THREE questions from a total of four questions, posed from two classes.

**Time:** 2 hours

## Math 640: Complex Variables

1. Let $f(z) = u(x,y) + iv(x,y)$ be analytic in domain $D \subseteq \mathbb{C}$ where $z = x + iy$. Prove that $u$ and $v$ are harmonic functions.

2. (a) Prove that the inverse of any Linear Fractional Transformation is also a Linear Fractional Transformation.

   (b) Find a Linear Fractional Transformation mapping $0 \to \infty,\ 1 \to i$, and $-1 \to 1$.

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## Math 660: Topology

The following will be used for Problems 3 and 4.

Define an alternative topology on $\mathbb{R}$, called the **upper limit topology** and denoted by $\mathbb{R}_{UL}$, generated by a **basis** consisting of all sets of the form

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

Define the set $E \subset \mathbb{R}_{UL}$ to be

$$E = (2, 5) = \{x \in \mathbb{R} \mid 2 < x < 5\}.$$

Recall

**Definition:** Let $(X, \mathcal{T})$ be a topological space, $A$ a subset of $X$, and $p$ a point in $X$. Then $p$ is a **limit point of** $A$ if and only if for each open set $U$ containing $p$, we have

$$(U - p) \cap A \neq \emptyset.$$

3. (a) Is 2 a limit point of the set $E$ in $\mathbb{R}_{UL}$? Prove your answer directly from the definition above.
   (b) Is the set $E$ open in $\mathbb{R}_{UL}$? Prove your answer.
   (c) Is the set $E$ closed in $\mathbb{R}_{UL}$? Prove your answer.
4. (a) Is the set $E$ compact? Prove your answer.
   (b) Is the set $E$ connected in $\mathbb{R}_{UL}$? Prove your answer.

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## Math 675: Differential Equations

5. (a) Find the general solution to

$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -4 \\ \frac{1}{4} & 7 \end{pmatrix} \mathbf{x}(t)$$

   (b) Find the solution to the IVP

$$\mathbf{x}'(t) = \begin{pmatrix} 5 & -4 \\ \frac{1}{4} & 7 \end{pmatrix} \mathbf{x}(t), \ \mathbf{x}(0) = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

6. Consider the ODE

$$(x-1)y'' + 3xy' - 8y = 0.$$

   (a) Verify that the point $x_0 = 1$ is a regular singular point.

   (b) Find the roots of the indicial equation. Let $r_1 =$ the root closest to $+\infty$ (the larger root) and $r_2 =$ the root closest to $-\infty$ (the smaller root).

   (c) For any solution(s) you find, calculate the first four nonzero terms explicitly.

   **One Solution:** If $r_1 - r_2 =$integer, then use the Frobenius method to find one solution corresponding to $r_1$, about the point $x_0 = 1$.

   **Two Solutions:** If the roots do not differ by an integer, then use the Frobenius method to find two independent solutions, $y_1$ and $y_2$, about the point $x_0 = 1$.

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## Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rl} \text{minimize} & 10x_1 + 9x_2 + 8x_3 + 8x_4 \\ \text{subject to} & 3x_1 + x_2 + 3x_3 + x_4 \leq 10 \\ & 5x_1 + 4x_2 + 2x_3 + 3x_4 \geq 18 \\ & 2x_1 + 3x_2 + 6x_3 + 4x_4 \geq 18 \\ & x_1 + 4x_2 + 3x_3 + 4x_4 \leq 18 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

8. Using the Complementary Slackness Theorem, determine whether $(3, 0, 2, 3)$ is the optimal solution to

$$\begin{array}{rl} \text{minimize} & 15x_1 + 5x_2 + 3x_3 + 5.5x_4 \\ \text{subject to} & 4x_1 - 3x_2 + 2x_3 + x_4 \geq 19 \\ & 5x_1 + x_2 - 2x_3 + 3x_4 \geq 20 \\ & 2x_1 - x_2 - 6x_3 - x_4 \leq -8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

**Department of Mathematics**
**Comprehensive Examination**
**2022 Spring Semester Core Classes**

**Time:** 2.5 hours **Directions:**

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

## Math 620

1. Consider $\mathbb{Z}$ as a group under addition.
   (a) Prove that all subgroups of $\mathbb{Z}$ are of the form $n\mathbb{Z} = \{nk \ : \ k \in \mathbb{Z}\}$ for $n \in \mathbb{Z}$.
   (b) Consider the group $\mathbb{Z}_2$ under addition modulo 2. Prove the quotient group $\mathbb{Z}/2\mathbb{Z}$ and the group $\mathbb{Z}_2$ are isomorphic as groups.
   (c) A subgroup $H$ of a group $G$ is said to be a *maximal subgroup* if there does not exist a proper subgroup $N$ of $G$ such that $H < N < G$. Prove $2\mathbb{Z}$ is a maximal subgroup of $\mathbb{Z}$.

2. Let $R$ and $S$ be commutative rings with "unity" (by which we mean a multiplicative identity) and let $\varphi : R \to S$ be a ring homomorphism.
   (a) Let $J$ be a subring of $S$. Prove the inverse image of $J$, $\varphi^{-1}(J) = \{r \in R : \varphi(r) \in J\}$, is a subring of $R$.
   (b) Let $J$ be a left ideal of $S$. Prove the inverse image of $J$, $\varphi^{-1}(J)$, is a left ideal of $R$.
   (c) Let $P$ be a prime ideal of $S$. Prove that *either* $\varphi^{-1}(P)$ is a prime ideal of $R$ *or* $\varphi^{-1}(P) = R$.

## Math 630

3. (a) Let $\langle M_1, \rho_1\rangle$ and $\langle M_2, \rho_2\rangle$ be metric spaces. Let $f : \langle M_1, \rho_1\rangle \to \langle M_2, \rho_2\rangle$ be a continuous function. Prove that if $K$ is compact in $M_1$, then the image $f(K)$ is compact in $M_2$.
   (b) Let $\mathbb{R}^1$ denote set $\mathbb{R}$ with the usual absolute value metric $\rho(x, y) = |x - y|$.
   Let $a < b$ be fixed real numbers and $[a, b] \subset \mathbb{R}^1$. Recall the following theorem from Calculus:

   **Theorem 1 (Extreme Value Theorem)** *If $f : [a, b] \to \mathbb{R}^1$ is continuous, then $f$ is bounded. Moreover, $f$ attains its maximum and minimum values.*

   Prove the Extreme Value Theorem.

Please turn over $\Longrightarrow$

4. Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} 3 & \text{if } x\in[0,1]\cap\mathbb{Q}\,, & \text{(that is, if } x\in[0,1] \text{ is rational)}\\ -2 & \text{if } x\in[0,1]\cap(\mathbb{R}\backslash\mathbb{Q})\,, & \text{(that is, if } x\in[0,1] \text{ is irrational)}\end{cases}$$

(a) Compute the upper and lower integrals

$$\overline{\int_0^1} f(x)dx\ , \qquad \text{and} \qquad \underline{\int_0^1} f(x)dx\ .$$

(b) Is $f$ Riemann integrable on $[0,1]$? Why or why not?

**Math 670**

5. Consider the equation

$$x=(x-3)^2-\cos x$$

(a) Prove that the equation has exactly two positive solutions.

(b) Let $\alpha$ be the smaller of the two solutions. Use Newton's Method to find an approximation of $\alpha$ with an absolute error of less than $10^{-5}$.

(c) Prove that your approximation is sufficiently accurate.

Note: for this problem, you may not use any graphing or root finding capabilities on your calculator.

6. Write an efficient algorithm to solve an $n\times n$ linear system $A\mathbf{x}=\mathbf{y}$ where the matrix $A$ has three nonzero diagonals as follows:

$$A=\begin{bmatrix} a_1 & 0 & d_3 & & & & & \\ b_1 & a_2 & 0 & d_4 & & & & \\ & b_2 & a_3 & 0 & d_5 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & b_{n-3} & a_{n-2} & 0 & d_n \\ & & & & & b_{n-2} & a_{n-1} & 0 \\ & & & & & & b_{n-1} & a_n \end{bmatrix},\ \mathbf{x}=\begin{bmatrix} x_1\\ x_2\\ x_3\\ \cdot\\ \cdot\\ \cdot\\ x_{n-2}\\ x_{n-1}\\ x_n\end{bmatrix},\ \mathbf{y}=\begin{bmatrix} y_1\\ y_2\\ y_3\\ \cdot\\ \cdot\\ \cdot\\ y_{n-2}\\ y_{n-1}\\ y_n\end{bmatrix}$$

Assume that $A$ is nonsingular and that pivoting is unnecessary.
You should first convert to an upper-triangular system and then apply back-substitution.
Fully exploit the sparsity pattern by writing everything in terms of the given vectors.
Note: your "algorithm" can be in "pseudo-code" rather than the proper syntax of a well-established programming language.