Department of Mathematics
Comprehensive Examination
2023 Spring Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of four questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

1. Find the number of zeros, counting multiplicities, of

$$f(z) = z^6 - 5z^4 + z^3 - 2z$$

inside the circle $\{z \in \mathbb{C} \,\big|\, |z| = 1\}$, and justify your conclusion.

2. Let f be an entire function such that there exists $M \in \mathbb{R}$ such that

$$|f(z)| \leq |z|^4 \text{ for each } |z| \geq M$$

Prove that f is a polynomial in z.

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Math 660: Topology

No Topology Exam was given in Spring 2023

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Math 675: Differential Equations

5. Find the solution about the point $x = 0$ to the following initial value problem. You only need to show values up through the x^4 term.

$$(x-1)y'' + x^2y' - 5y = 0,\ y(0) = 2,\ y'(0) = -1$$

6. Find the solution to

$$\mathbf{x}' = \begin{pmatrix} 8 & 5 \\ 3 & 6 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ e^{-t} \end{pmatrix}$$

Show all work. No technology is allowed except for basic arithmetic involving fractions.

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Math 680: Optimization

7. Solve the following problem using the Simplex method. Clearly show the set up of the problem, the Simplex tableau, and the solution.

$$\begin{array}{rllll} \text{minimize} & 4x_1 & +\,5x_2 & +\,3x_3 & +\,5 \\ \text{subject to} & 2x_1 & +\,5x_2 & +\,6x_3 & \leq 30 \\ & 4x_1 & +\,3x_2 & +\,4x_3 & \geq 36 \\ & 4x_1 & +\,2x_2 & +\,3x_3 & \geq 26 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Solve the following transportation problem, where the supplies are listed along the first column, and the demands are listed along the first row, and the costs of each route are contained in the table. Please use either Vogel's Method or the Column-by-Column Method as your starting point.

	40	15	30	20
35	4	8	10	12
10	9	7	7	10
35	6	3	3	2
25	7	6	5	4

Department of Mathematics
Comprehensive Examination
2023 Spring Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G be a group, and $\{H_i\}_{i\in I}$ a family of subsets of G.
   (a) If H_i is a subgroup for all $i \in I$, prove $\cap_{i\in I} H_i$ is a subgroup of G.
   (b) If H_i is a **normal subgroup** for all $i \in I$, prove that $\cap_{i\in I} H_i$ is a normal subgroup of G.
   (c) Provide a counterexample to show that if H_i are subgroups for all $i \in I$, then $\cup_{i\in I} H_i$ is not necessarily a subgroup of G. Write a justification to explain why your counterexample shows the claim is not true.

2. Consider the integers $\mathbb{Z}$ as a ring under addition and multiplication. Let $n \in \mathbb{Z}$ and consider $\mathbb{Z}_n$ as a ring under addition and multiplication modulo n. Define the map

$$\varphi : \mathbb{Z} \to \mathbb{Z}_n$$

   as $\varphi(a) = a \bmod n$. Assume this map was already shown to be a ring homomorphism.
   (a) Prove $\ker \varphi = \langle n \rangle$ where $\langle n \rangle$ is the ideal generated by n, i.e. $\langle n \rangle = \{n \cdot r : r \in \mathbb{Z}\}$.
   (b) Use the first isomorphism theorem of rings to prove $\mathbb{Z}/\langle n \rangle \cong \mathbb{Z}_n$.
   (c) What must be true of $n \in \mathbb{Z}$ in order for the ideal $\langle n \rangle$ to be a maximal ideal in $\mathbb{Z}$? Provide an explanation or justification of your conclusion using the isomorphism given in (b).

Math 630

3. (a) Let $U_1, U_2, U_3, \ldots$ be open subsets of a metric space (M, d). Prove that the union

$$\bigcup_{i=1}^{\infty} U_i$$

   is open in M.You should justify (prove) any property you use beyond the definition of an "open" set.
   (b) Give an example to illustrate why the intersection of infinitely many open sets need not be open.

4. Let $f_n(x) = \frac{x^n}{2+x^n}$.
   (a) Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0, 1/3]$.
   (b) Does $\{f_n\}_{n=1}^{\infty}$ converge uniformly on $[0, 1]$? Justify your claim.

⇐ Turn Over ⇒

5. Consider the equation
$$\ln(x) = (x-2)^2 - 3.$$
   (a) Prove that the equation has exactly two solutions.
   (b) Let α be the larger of the two solutions and use Newton's Method to find an approximation of α with an absolute error less than 10^{-5}.

6. Consider $A = (a_{ij})$ and its splitting $A = D - L - U$ where
$$D = \mathrm{diag}(a_{11}, a_{22}, .., a_{nn}),\ a_{ii} \neq 0$$
is the diagonal part of A, and
$$-L = \begin{bmatrix} 0 & & & \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & 0 \end{bmatrix}, \quad -U = \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ & \ddots & & \vdots \\ & & 0 & a_{n-1,n} \\ & & & 0 \end{bmatrix}$$
are the lower and upper triangular parts of A respectively. Consider the following linear system:
$$\begin{array}{rcrcrl} 4x_1 & - & x_2 & - & x_3 & = 3 \\ -2x_1 & + & 6x_2 & + & x_3 & = 9 \\ -x_1 & + & x_2 & + & 7x_3 & = -6 \end{array}, \quad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
   (a) For the linear system above, starting with $\mathbf{x}^{(0)}$, find $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ using the Jacobi method.
   (b) Will the Jacobi method converge for any initial vector?