Department of Mathematics
Comprehensive Examination
2025 Spring Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

1. (a) State Liouville's theorem.

   (b) Prove the following statement: If f is entire such that

   $$|f(z)| \leq |e^z|, \qquad \forall z \in \mathbb{C} \ ,$$

   then

   $$f(z) = ce^z \text{ for some } c \in \mathbb{C} \text{ s.t. } |c| \leq 1 \ .$$

2. Let

   $$T(z) = \frac{az+b}{cz+d}$$

   be a linear fractional transformation that is not the identity map. Prove that T has at most two fixed points.

3. Use the method of residues to evaluate

   $$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

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Math 660: Topology

4. (Note: for this exercise you may **not** use the Heine-Borel Theorem or any equivalent statement). Define the set H to be:

$$H = \left\{ \frac{1}{n} \mid n = 1, 2, 3, ... \right\} = \left\{ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}, ... \right\}$$

   Consider $H \subset \mathbb{R}_{std}$.

   (a) Find an open cover of H that has no finite subcover (in $\mathbb{R}_{std}$). Provide a full, detailed proof to support your assertion.

   (b) Consider the set
   $$\overline{H} = H \cup \{0\}$$
   Give a full, detailed proof of the fact that every open cover of $\overline{H}$ has a finite subcover (in $\mathbb{R}_{std}$).

5. Let $(X, \mathcal{T})$ be a topological space, and let U be an open set and A be a closed subset of X.

   (a) Prove that the set $U - A$ is open. You need to provide a **full proof** of this fact directly from definitions.

   (b) Prove that the set $A - U$ is closed. You need to provide a **full proof** of this fact directly from definitions.

6. Recall the following definitions discussed in class:

   Definition: Let $\mathbb{Z}_{arith}$ be the set $\mathbb{Z}$ with a topology whose basis elements are **arithmetic progressions**, i.e., sets of the form
   $$\{az + b : z \in \mathbb{Z}\}, \text{ for } a, b \in \mathbb{Z}, a \neq 0.$$

   Definition: Let $(X, \mathcal{T})$ be a topological space. X is **Hausdorff**, or a T_2**-space**, if and only if for every pair x, y of distinct points there are disjoint open sets U, V such that $x \in U$ and $y \in V$.

   Give a full, detailed proof of the fact that $\mathbb{Z}_{arith}$ is a Hausdorff space.

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Math 675: Differential Equations

7. (a) Find the general solution for the system

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{x}(t)$$

   (b) Sketch a few solutions (by hand) and explain how you know your sketch is correct.

   (c) What's the behavior of solutions $\mathbf{x}(t)$ as $t \to +\infty$?

   Notes for part c)

   - You must cover all possible cases – that is, you have to handle all possible sets of initial conditions.
   - If there's a "dominant direction line" you must specify whether solutions asymptote to this line or whether they become parallel to this line on the phase plane.

8. Consider the matrix:

$$A = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$$

The eigenvalues for A are $\lambda_1 = -4$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v}_{\lambda_1} = (-1, 1)$ and $\mathbf{v}_{\lambda_2} = (1, 2)$. Use this information to find the general solution to the inhomogeneous system:

$$\mathbf{x}'(t) = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \mathbf{x}(t) + e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

9. Consider the following "rabbits vs. sheep" problem (a 2D nonlinear system) where $x, y \geq 0$.

$$x' = x(4 - 2x - y), \qquad y' = y(3 - x - y)\ .$$

Do all of the following:

   (a) Find all fixed points

   (b) Investigate the stability of each fixed point.

Department of Mathematics
Comprehensive Examination
2025 Spring Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620

1. Let G and H be groups, and $\varphi : G \to H$ a group homomorphism.
   (a) Let K be a subgroup of H. Prove that $\varphi^{-1}(K) = \{g \in G : \phi(g) \in K\}$ is a subgroup of G.
   (b) Recall, N is a *normal subgroup* of a group G if N is a subgroup and $gNg^{-1} = N$ for all $g \in G$. Assume that we have already proved that $\ker \varphi = \{g \in G : \phi(g) = e_G\}$ is a subgroup of G. Prove that $\ker \varphi$ is a normal subgroup of G.
   (c) Assuming that $\ker \varphi$ is a normal subgroup, prove that, if J is any subgroup of G, then $\ker \varphi \cap J$ is a normal subgroup of J.
2. Let $\varphi : R \to S$ be a surjective homomorphism of rings. Consider the center Z of the ring R, i.e. the set $Z(R) = \{z \in R : zr = rz \text{ for all } r \in R\}$
   (a) Prove that $Z(R)$ is a subring of the ring R.
   (b) Prove that the image of the center of R is contained in the center of S, i.e. $\varphi(Z(R)) \subseteq Z(S)$.
3. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ; a, b, d \in \mathbb{Z} \right\}$ be a subring of $M_{2\times 2}(\mathbb{Z})$, where $M_{2\times 2}(\mathbb{Z})$ is a ring under the operations of matrix addition and matrix multiplication. Consider the map

$$\varphi : R \to \mathbb{Z} \times \mathbb{Z}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d)$$

   where $\mathbb{Z} \times \mathbb{Z} = \{(x, y) : x, y \in \mathbb{Z}\}$ is a ring under component-wise addition and multiplication.
   (a) Prove φ is a ring homomorphism.
   (b) Show that φ is surjective.
   (c) Prove that $\ker \varphi = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} ; b \in \mathbb{Z} \right\}$.

4. Let h be a continuous function on $(-\pi, \pi]$ and periodically and continuously extended to $\mathbb{R}$. Prove that h is uniformly continuous on $\mathbb{R}$.

5. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed vector spaces. Let $T : X \to Y$ be a linear map.
   (a) What does it mean for T to be bounded? (State the definition of a bounded linear operator).
   (b) Prove that if T is bounded, then T is continuous.

6. Let $f, g \in L^2([a, b])$. Prove that

$$\int_a^b |f(x)g(x)|\ dx \leq \|f\|_2 \|g\|_2$$

where $\|h\|_2 = \left(\int_a^b |h(x)|^2\ dx\right)^{1/2}$.

7. (a) State the conditions on the function g such that the fixed-point sequence defined by $p_{n+1} = g(p_n)$ converges to a unique fixed point p.
   (b) Let $g(x) = 2 - e^{-x}$ and prove that $x = g(x)$ has exactly two real solutions $\alpha_1 < \alpha_2$.
   (c) Consider approximating the larger solution, α_2, by applying a fixed-point iteration on $g(x)$. Determine an interval $[a, b]$ such that the fixed-point sequence defined by $p_{n+1} = g(p_n)$ is guaranteed to converge to α_2 for any initial approximation $p_0 \in [a, b]$. Justify your answer using the conditions from part (a).
   (d) Can you find an interval for the smaller solution α_1? Why or why not?
   (e) In the graph on a separate page, $g(x)$ is plotted along with the line $y = x$. Draw a fixed point iteration scheme (i.e. sketch the cobweb/boxes) corresponding to each choice of p_0 indicated.

8. Let $A = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \\ 12 \end{pmatrix}$.
   (a) Factor the matrix A using your choice of factorization method.
   (b) Utilize the factorization obtained in (a) to solve the system $Ax = b$.

9. Given the linear system $Ax = b$ where

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

   (a) What relevant properties does the matrix A have?
   (b) Will the Jacobi and/or the Gauss-Seidel methods converge for this linear system? Why or why not?
   (c) Find the first iteration using the Jacobi method with $x^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
   (d) Find the first iteration using the Gauss-Seidel method with $x^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

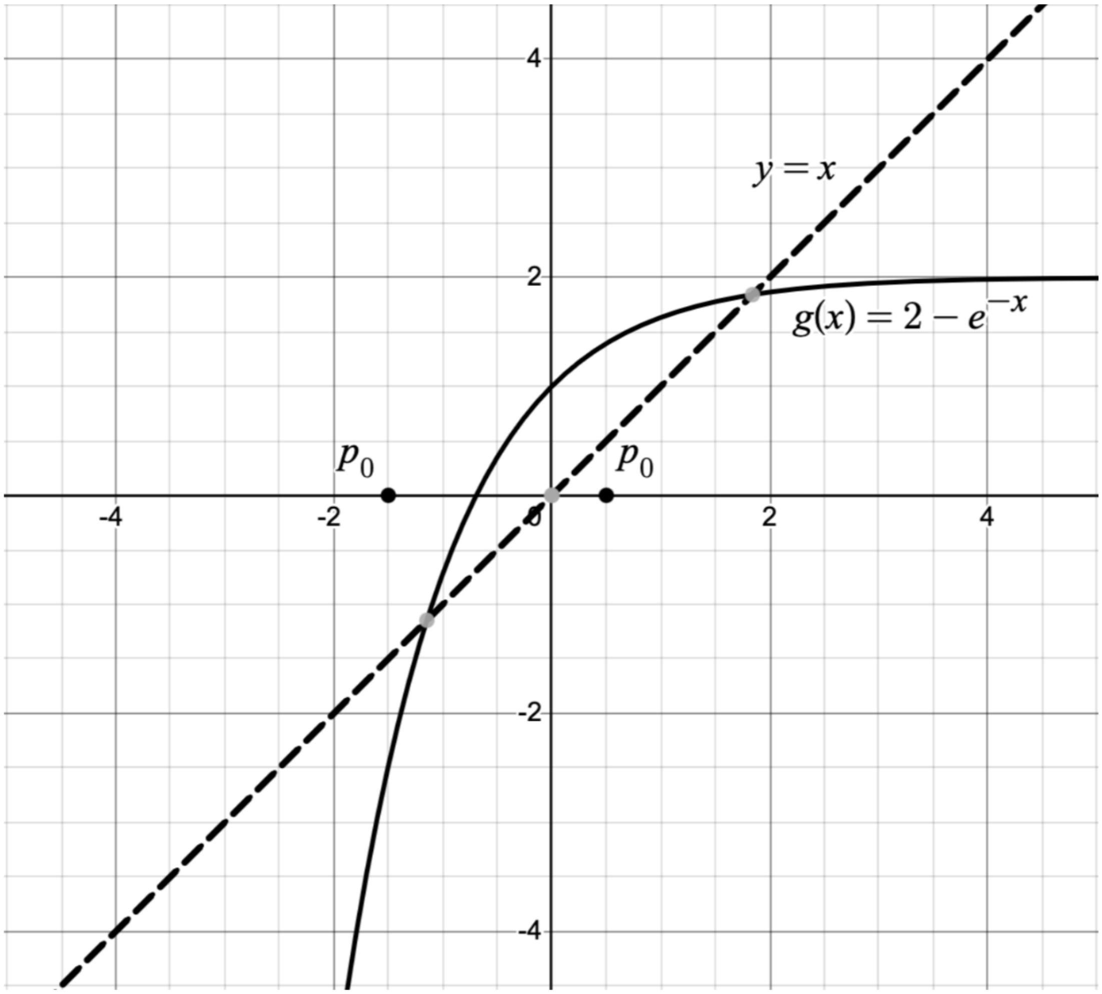

Figure 1: $g(x) = 2 - e^{-x}$