Department of Mathematics
Comprehensive Examination
2026 Spring Semester "Choose 2" Classes

Directions: You will answer THREE questions from a total of six questions, posed from two classes.

Time: 2 hours

Math 640: Complex Variables

1. Suppose that $f(z)$ is entire and that the harmonic function

$$u(x, y) = \mathrm{Re}[f(z)]$$

has an upper bound u_0, such that $u(x, y) \leq u_0, \ \forall (x, y)$.

Prove that $u(x, y)$ must be constant throughout the plane.
Hint: Apply Liouville's Theorem to $g(z) = e^{f(z)}$.

2. Find the Linear Fractional Transformation

$$T(z) = \frac{az + b}{cz + d}$$

which maps $-1, 1, \infty$ to $1, \infty, -1$ respectively.

3. Find all possible values of

$$\oint_\Gamma \frac{(z+1)^2}{(z-3)}\, dz$$

where Γ is a positively oriented simple closed curve in $\mathbb{C}$ not passing through the point 3. Make sure to address how you can get different answers.

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Math 660: Topology

No Topology Exam was given in Spring 2026.

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Math 675: Differential Equations

4. Consider the system

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & 1 \\ 3 & -3 \end{pmatrix} \mathbf{x}(t).$$

The general solution to this system is given by:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

a) Use the given general solution to find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. **You do not have to compute these from scratch.**

b) Classify the origin (node, saddle, spiral, etc.).

c) Suppose a solution satisfies the initial condition $x(0) = 3, y(0) = 3$. What are the constants c_1, c_2 corresponding to this initial condition? What is the slope $\dfrac{dy}{dx}$ of this solution curve? Justify your answer.

d) Describe the geometric behavior of trajectories as $t \to +\infty$. Do they become tangent to one of the eigendirection lines? If so, which one? Be sure to address all possible initial conditions.

e) Now do the same analysis backward in time: Describe the geometric behavior of trajectories as $t \to -\infty$. Describe the geometric behavior of trajectories as $t \to -\infty$. Do they become parallel to one of the eigendirection lines? If so, which one? Be sure to address all possible initial conditions.

f) As $t \to -\infty$, do trajectories converge to one of the eigendirection lines? Why or why not?

g) Sketch the phase portrait by hand and explain how you know your sketch is correct.

5. Consider the system

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \mathbf{x}(t).$$

a) Find the eigenvalues of the coefficient matrix.

b) Classify the fixed point at the origin: is it a stable spiral, unstable spiral, or center? Explain how you know.

c) Find the general solution to the system. **Express your answer in terms of real-valued functions**.

d) Describe the direction of rotation in the phase portrait (clockwise or counterclockwise). Explain briefly.

6. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$ and $\lambda_2 = 1$, with corresponding eigenvectors

$$\mathbf{v}_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_{\lambda_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

a) Calculate $\exp(t\mathbf{A})$.

b) Use part (a) to solve the IVP

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{x}(t), \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

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Math 680: Optimization

7. Prove or disprove, using the Complementary Slackness Theorem:
(3,1,4) is the optimal solution to the following linear programming problem.

$$\begin{array}{rrrrl} \text{Maximize} & 10x_1 & + 12x_2 & + 8x_3 & \\ \text{Subject to} & 3x_1 & + 5x_2 & + x_3 & \leq 22 \\ & 2x_1 & - x_2 & + 4x_3 & \leq 21 \\ & x_1 & + 2x_2 & + 6x_3 & \leq 29 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

8. Solve the following minimization problem using the Simplex method. Please show all your work.

$$\begin{array}{rrrrl} \text{minimize} & -3x_1 & + 2x_2 & + 5x_3 & \\ \text{subject to} & -x_1 & - 4x_2 & + 3x_3 & \geq 15 \\ & 2x_1 & + x_2 & + 4x_3 & \geq 18 \\ & 5x_1 & + x_2 & + 2x_3 & \leq 30 \\ & & & x_j & \leq 0 \end{array}$$

9. Consider the problem

$$\begin{array}{rrrrl} \text{Maximize} & 5x_1 & + 8x_2 & + 9x_3 & \\ \text{Subject to} & 2x_1 & + x_2 & + x_3 & \leq 2 \\ & 4x_1 & + 2x_2 & + 3x_3 & \leq 3 \\ & x_1 & + 3x_2 & + 3x_3 & \leq 4 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

Below is the first and last tableau in the Simplex method solution of this maximization problem.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	2	1	1	1	0	0	2
x_5	4	2	3	0	1	0	3
x_6	1	3	3	0	0	1	4
	-5	-8	-9	0	0	0	0
x_4	$\frac{5}{3}$	0	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
x_3	$\frac{10}{3}$	0	1	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
x_2	-3	1	0	0	-1	1	1
	1	0	0	0	1	2	11

For each of the following scenarios return to the original problem. Use sensitivity analysis to answer each question.

(a) What is the range on the coefficient of x_2 in the objective function such that the basis variables do not change?

(b) Find the new solution if the right hand side constant in the third constraint changes from 4 to 6.

(c) Add the constraint

$$2x_1 + 2x_2 + 3x_3 \leq 2$$

Department of Mathematics
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2026 Spring Semester Core Classes

Time: 2.5 hours
Directions:

- You will answer FOUR questions.
- You must choose at least ONE question from each class (one from Math 620, one from Math 630, and one from Math 670).
- The fourth question can come from any of the classes.

Math 620 Applied Linear Algebra

1. Suppose a sequence of numbers is described by the linear recurrence relation

$$x_n = 4x_{n-1} - x_{n-2}, \text{ for } n = 2, 3, \ldots, \text{ with starting values } x_0 = 0,\ x_1 = 1$$

   (a) Find the next two terms in the sequence, x_2 and x_3.

   (b) Find the n^{th} term in the sequence explicitly, without having to find the first $(n-1)$ terms, using the concept of matrices.

2. Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$.

   (a) Prove that, for any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{v}$.

   (b) Prove that if matrix A is invertible, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{v}$.

3. Suppose a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the following properties:

$$T\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} -9 \\ 12 \end{pmatrix} \quad T\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -11 \\ -13 \end{pmatrix}$$

   (a) Determine the matrix A associated with the linear transformation T.

   (b) Determine the stretch factors and stretch directions of T.

4. Assume f can be represented as a trigonometric series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Explain how we derive formulas for the coefficients: a_0, a_n, b_n for $n \geq 1$. Your solution should give one complete derivation (for a_0, a_n, or b_n) and at least a description of how you would derive the others (and what you would get).

These trig identities may be helpful:

(a) For all $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} \cos(nx)dx = 0$ and $\int_{-\pi}^{\pi} \sin(nx)dx = 0$.

(b) For all $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} \cos^2(nx)dx = \pi$ and $\int_{-\pi}^{\pi} \sin^2(nx)dx = \pi$.

(c) For all $m, n \in \mathbb{N}$, $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0$.

(d) For all $m, n \in \mathbb{N}$ with $m \neq n$, $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0$ and $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0$.

5. Let $h(x) = \sum_{n=1}^{\infty} \frac{1}{3x^2 + n^3}$. Prove that h is a continuous function on $\mathbb{R}$.

6. Consider the space of bounded sequences of real numbers:

$$\ell^\infty = \{\{x_n\} : \sup\{|x_n| : n \in \mathbb{N}\} < \infty\}$$

with norm $\|\{x_n\}\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$. Prove that ℓ^∞ with this norm is complete.

(You can assume that ℓ^∞ is a vector space and that $\|\cdot\|_\infty$ is a norm. You may also assume results from undergraduate real analysis, but you should state explicitly what result you are using.)

Math 670 Numerical Analysis

7. (a) Define quadratic convergence.

   (b) Suppose

   - f is defined on $[a, b]$
   - f' and f'' exist and are continuous on (a, b)
   - There exists $p \in (a, b)$ such that $f(p) = 0$
   - $f'(p_n) \neq 0$
   - $|f''(x)| < M$ on (a, b)

   Derive the error formula for Newton's method

   $$|p - p_{n+1}| \leq \frac{M}{2\,|f'(p_n)|}\,|p - p_n|^2$$

   Hint: Start with the Taylor series expansion of $f(p)$ about the point p_n.

8. Let $A = \begin{pmatrix} 2 & -2 & -3 \\ -2 & 5 & 4 \\ -3 & 4 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ -12 \\ -12 \end{pmatrix}$.

   (a) Factor the matrix A using your choice of factorization method.

   (b) Utilize the factorization obtained in (a) to solve the system $Ax = b$.

9. Prove that if A is strictly row diagonally dominant, then the Jacobi method converges for any choice of the initial approximation $x^{(0)}$.

   Hint: If you are not sure where to start, show why it's true for an example such as

   $$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

   What does the matrix $B_J = D^{-1}(L+U)$ look like in this case? Then make an argument why $||B_J||_\infty < 1$. Can you generalize this result?