Problem for 1996 March

Proposed by Professor Stu Smith

For $n=1,2,3,\ldots$ let X_n be the continued fraction as follows.

$$X_n = a_1 + \cfrac{1}{a_2 + \cfrac{\strut 1}{a_3 + \cfrac{\strut 1}{a_4 + \cfrac{\strut 1}{\ddots + \cfrac{\strut}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

Show that when X_n is written as a ``proper fraction'' (i.e., no fraction in numerator or denominator) the number of terms in both the numerator and the denominator are Fibonacci numbers (i.e., elements of the sequence $0,1,1,2,3,5,8,13,21,\ldots$).

For example,

$$X_4 = \frac{a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1}{a_2a_3a_4 + a_2 + a_4},$$

in which the numerator consists of 5 terms and the denominator consists of 3 terms.