Problem for 1996 August

Proposed anonymously

From an observation by Vladimir Arnol'd

Determine whether

$$\lim_{x\to 0} \frac{\tan(\sin x)-\sin(\tan x)}{\arctan(\arcsin x)-\arcsin(\arctan x)}$$

exists; if it exists, determine its value.

Solution by Dan Jurca

We show the limit exists and equals 1.

One can find these well-known expansions in various tables:

$$\begin{aligned}
\tan x &= x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\dots\\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\dots\\
\arctan x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\dots\\
\arcsin x &= x+\frac{1\cdot x^3}{2\cdot 3}+\frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7\cdot x^9}{2\cdot 4\cdot 6\cdot 8\cdot 9}+\dots
\end{aligned}$$

By the analyticity of each of these functions at 0, and by the analyticity of the composition of analytic functions, we may substitute (for example) the series for sinx into the series for tanx and obtain

$$\tan(\sin x))=x+\frac{x^3}{6}-\frac{x^5}{40}-\frac{107x^7}{5040}-\frac{73x^9}{24192}+\dots$$

(The solver wrote a computer program to do the (tedious) calculations.)

Similarly

$$
\begin{aligned}
\sin(\tan x)) &= x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{55x^7}{1008} - \frac{143x^9}{3456} + \dots \\
\arctan(\arcsin x)) &= x - \frac{x^3}{6} + \frac{13x^5}{120} - \frac{173x^7}{5040} + \frac{12409x^9}{362880} + \dots \\
\arcsin(\arctan x)) &= x - \frac{x^3}{6} + \frac{13x^5}{120} - \frac{341x^7}{5040} + \frac{18649x^9}{362880} + \dots
\end{aligned}
$$

Substituting these expansions we find that for x near to (but different from) 0:

$$
\begin{aligned}
\frac{\tan(\sin x) - \sin(\tan x)}{\arctan(\arcsin x) - \arcsin(\arctan x)} &= \frac{\frac{1}{30}x^7 + \frac{29}{756}x^9 + \dots}{\frac{1}{30}x^7 - \frac{13}{756}x^9 + \dots} \\
&= 1 + \frac{5}{3}x^2 + \dots
\end{aligned}
$$

so that the limit exists and equals 1, as asserted.