Problem for 1997 April

Communicated by Professor Bill Nico
From an observation by Ronald Graham

$$\sqrt{5}+\sqrt{22+2\sqrt{5}} = \sqrt{11+2\sqrt{29}} + \sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}$$

Solution by Dan Jurca

With $u=\sqrt{11+2\sqrt{29}}$ and $v=\sqrt{11-2\sqrt{29}}$ we find $uv=\sqrt{11^2-4\cdot 29}=\sqrt{5}$. Hence

$$
\begin{aligned}
22+2\sqrt{5} &= (11+2\sqrt{29})+2\sqrt{5}+(11-2\sqrt{29}) \\
&= u^2+2uv+v^2 \\
&= (u+v)^2. \text{ Hence, since } 0 < u+v, \\
\sqrt{22+2\sqrt{5}} &= u+v \\
&= u+\frac{\sqrt{5}}{u}. \text{ Therefore} \\
\sqrt{5}+\sqrt{22+2\sqrt{5}} &= u+\sqrt{5}+\frac{\sqrt{5}}{u} \\
&= u+\frac{(u+1)\sqrt{5}}{u} \\
&= u+\sqrt{\frac{5(u+1)^2}{}}
\end{aligned}
$$

$$\sqrt{\quad} \qquad u^2$$

$$=u+\sqrt{\frac{5u^2+10u+5}{u^2}}$$

$$=u+\sqrt{5+\frac{5}{u^2}+\frac{10}{u}}$$

$$=u+\sqrt{5+v^2+2\sqrt{5}\cdot\frac{\sqrt{5}}{u}}$$

$$=u+\sqrt{5+v^2+2\sqrt{5}\cdot v}$$

$$=\sqrt{11+2\sqrt{29}}+\sqrt{5+11-2\sqrt{29}+2\sqrt{5}\sqrt{11-2\sqrt{29}}}$$

$$=\sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}\ ,$$

as asserted.