Problem for 1998 November

Communicated by Dan Jurca

Let $f(x)=\sin(\pi \sin^{-1}(x))$; for n=0,1,2,3,… evaluate the n-th derivative of f at 0; *i.e.*, compute

$$f^{(n)}(0).$$

Solution by Dan Jurca

Let us write $y=f(x)=\sin(\pi \sin^{-1}x)$; then we find

$$\begin{aligned} y &= \sin(\pi \sin^{-1}x) \\ \sin^{-1}y &= \pi \sin^{-1}x \\ \frac{y'}{\sqrt{1-y^2}} &= \frac{\pi}{\sqrt{1-x^2}} \\ (1-x^2)(y')^2 &= \pi^2(1-y^2) \\ -2x(y')^2+(1-x^2)\cdot 2y'y'' &= \pi^2\cdot -2yy' \\ -xy'+(1-x^2)y'' &= -\pi^2 y, \end{aligned}$$

so that y satisfies the differential equation $(1-x^2)y''-xy'+\pi^2y=0$. Since f is clearly an analytic function near 0, we write $y=\sum_{n=0}^{\infty} c_n x^n$, where $c_n=[(f^{(n)}(0))/n!]$. We determine c_n using standard techniques as follows.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$\pi^2 y = \sum_{n=0}^{\infty} \pi^2 c_n x^n$$

$$y' = \infty \quad (n+1)c_{n+1}x^n$$

$$\sum_{n=0}$$

$$xy' = \sum_{n=0}^{\infty} nc_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

$$x^2y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^n$$

$$(1-x^2)y'' = \sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2}-n(n-1)c_n]x^n$$

Hence for n=0,1,2,… we have

$$(n+2)(n+1)c_{n+2}-n(n-1)c_n-nc_n+\pi^2c_n=0.$$

Now one finds at once $c_0=f(0)=0$, $c_1=f'(0)=\pi$, and

$$2 \le n \Rightarrow c_n=\frac{(n-2)^2-\pi^2}{n(n-1)}\,c_{n-2}.$$

It follows that

$$f^{(n)}(0)=n!\cdot c_n=\begin{cases} 0 & \text{if } n \text{ is even;} \\ \pi & \text{if } n=1; \\ (1^2-\pi^2)(3^2-\pi^2)\ldots[(n-2)^2-\pi^2]\pi & \text{if } n \text{ is odd and } 3 \le n. \end{cases}$$

Equivalently one may write

$$f^{(n)}(0)=\begin{cases} 0 & \text{if n is even;} \\ \pi \prod_{i=1}^{\lfloor n/2 \rfloor} [(2i-1)^2-\pi^2] & \text{if n is odd.} \end{cases}$$