Problem for 1998 December

Communicated by Dan Jurca

Let us define a **box** as a subset of the cartesian plane which is the interior of a square with sides of length 1 and vertices at the integer lattice points; *i.e.*, each corner of the square has integral coordinates. Now consider positive integers m and n, and the m×n rectangle with one corner at the origin of the cartesian plane and the diagonally opposite corner at the point (m,n). Find a formula for b(m,n), the number of boxes inside this rectangle which are intersected by the line from the point (0,0) to the point (m,n).

From the sketch below we can see that b(4,7)=10, and b(4,8)=8.

Solution by Dan Jurca

We show that b(m,n)=m+n–gcd(m,n).

First consider the case that m and n are relatively prime, so that gcd(m,n)=1. Then we show that the line segment l from (0,0) to (m,n) does not contain a point (i,j) with 0 < i < m. For otherwise the triangles with vertices (0,0),(i,0),(i,j) and (0,0),(m,0),(m,n) are similar, so that i/j=m/n. But then we have in=jm, so that m|in (since m divides jm). Since m and n are relatively prime, it follows that m|i. This is not possible if 0 < i < m. Hence l contains no other point with integer coordinates. This means that if l intersects the boundary of a box, it does so in one of precisely three ways:

1. in the point (0,0);
2. in the *interior* of an edge of the boundary;
3. in the point (m,n).

Now l cuts m–1 vertical edges and n–1 horizontal edges. Thus there is one intersection of type 1; there are m+n–2 intersections of type 2; and there is one intersection of type 3. Hence there are m+n intersections of all three types. Consider a point, say P, moving along l from (0,0) to (m,n). Immediately after each intersection except the last one, the one at (m,n), P enters a box. Thus P enters m+n–1 boxes, and therefore l intersects exactly m+n–1 boxes.

Now consider the general case, and let g=gcd(m,n). Then there are positive integers m′ and n′ such that m=gm′, n=gn′, and gcd(m′,n′)=1. Again considering similar triangles the line segment l from (0,0) to (m,n) contains the point (m′,n′). Further, the pattern from (0,0) to (m′,n′) repeats g times. Hence l intersects $g(m'+n'-1) = gm'+gn'-g=m+n-\gcd(m,n)$ boxes, so $b(m,n)=m+n-\gcd(m,n)$, as asserted.

Also solved by Matthew Hubbard, Thomas Kim, and Professor Bill Nico. Thomas Kim generalized the result to higher dimensions.

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