Problem for 2000 February

Proposed by Dan Jurca

Prove that if $z \in \mathbf{C}$ and $1 < |z|$, then

$$\sum_{n=1}^{\infty} \frac{1}{z^n+1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n-1} = 0.$$

Solution by the proposer

We shall prove a more general result. Precisely, we show

$$z \in \mathbf{C} \text{ and } 1 < |z| \Rightarrow \sum_{n=1}^{\infty} \frac{1}{z^n+1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n-1} = 0.$$

For if $1 < |z|$, then $1 \le d \Rightarrow$

$$\begin{aligned}
\frac{1}{z^d+1} &= \frac{z^{-d}}{1+z^{-d}} \\
&= \frac{z^{-d}}{1-(-z^{-d})} \\
&= z^{-d}-z^{-2d}+z^{-3d}-z^{-4d}+-\ldots \\
&= -\sum_{q=1}^{\infty} (-1)^q z^{-dq}, \quad \text{and} \\
\frac{(-1)^d}{z^d-1} &= (-1)^d \frac{z^{-d}}{1-z^{-d}} \\
&= (-1)^d (z^{-d}+z^{-2d}+z^{-3d}+z^{-4d}+\ldots) \\
&= (-1)^d \quad \infty \quad z^{-dq}.
\end{aligned}$$

$$\sum_{q=1}$$

Hence $1 < |z| \Rightarrow$

$$\sum_{n=1}^{\infty} \frac{1}{z^n+1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n-1} = \sum_{d=1}^{\infty} \frac{1}{z^d+1} + \sum_{d=1}^{\infty} \frac{(-1)^d}{z^d-1}$$

$$= \sum_{d=1}^{\infty} \left[-\sum_{q=1}^{\infty} (-1)^q z^{-dq} + (-1)^d \sum_{q=1}^{\infty} z^{-dq} \right]$$

$$= \sum_{d=1}^{\infty} \sum_{q=1}^{\infty} \left[(-1)^d - (-1)^q \right] z^{-dq}$$

$$= \sum_{n=1}^{\infty} \sum_{d|n} \left[(-1)^d - (-1)^{n/d} \right] z^{-n}$$

$=0$, since

$$a \in \mathbf{C},\ 1 \le n \Rightarrow \sum_{d|n} (a^d - a^{n/d}) = 0.$$

Also solved by Ed Keller, Massoud Malek, and John M. Sayer.