Problem for 2000 May

proposed by Dan Jurca

a.

Prove that if n is an integer and $2 \le n$, then

$$\frac{1}{2} < \left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot\left(1-\frac{1}{16}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right).$$

b.

Determine

$$\lim_{n\to\infty}\left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot\left(1-\frac{1}{16}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right).$$

Solution by the proposer

For $2 \le n$ write

$$p_n = \left(1-\frac{1}{4}\right)\cdot\left(1-\frac{1}{9}\right)\cdot\left(1-\frac{1}{16}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right);$$

then we show by induction that

$$2 \le n \Rightarrow p_n = \frac{1}{2} + \frac{1}{2n}.$$

Since $p_2 = 1 - {}^1/_4 = {}^3/_4 = {}^1/_2 + {}^1/_4 = {}^1/_2 + [\,1/2\cdot 2]$, the assertion certainly holds if n=2.

Next assume $3 \le n$, and $p_{n-1} = {}^1/_2 + [\,1/(2(n-1))]$. Then

$$\begin{aligned} p_n &= p_{n-1}\cdot\left(1-\frac{1}{n^2}\right) \\ &= \left(\frac{1}{2}+\frac{1}{2n-2}\right)\cdot\frac{n^2-1}{n^2} \end{aligned}$$

$$= \frac{1}{2} \cdot \left(1 + \frac{1}{n-1} \right) \cdot \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{1}{2} \cdot \frac{n}{n-1} \cdot \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{1}{2} \cdot \frac{n+1}{n}$$

$$= \frac{1}{2} + \frac{1}{2n},$$

so the assertion holds by induction on n.

Part a. now follows at once, and clearly $\lim_{n\to\infty} p_n = {}^1/_2$; in fact $(p_n)_{n=2}^{\infty} \downarrow {}^1/_2$.

Also solved by Matthew Hubbard, John M. Sayer, Dr. Louis Villanueva, and Professor Bill Nico