Problem for 2000 June

proposed by Dan Jurca

Recall the following notation.

$$1 \le N \Rightarrow H_N = \sum_{n=1}^{N} \frac{1}{n},$$

the N-th partial sum of the harmonic series, and let $H_0=0$.

Now suppose that a and b are integers, and that $0 < a < b$.

a.

Prove that

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{b-a}\left(H_{b-1} - H_{a-1} \right).$$

b.

Call the sum above $S(a,b)$; prove that there exists a polynomial $p(x)$ with integer coefficients and degree $b-a$ such that

$$0 \le N \Rightarrow \sum_{n=0}^{N} \frac{1}{(n+a)(n+b)} = S(a,b) - \frac{1}{b-a} \cdot \frac{p'(N)}{p(N)},$$

where p' is the derivative of p.

Solution by the proposer

$$0 \le N \Rightarrow \sum_{n=0}^{N} \frac{b-a}{(n+a)(n+b)} = \sum_{n=0}^{N} \left(\frac{1}{n+a} - \frac{1}{n+b} \right)$$

$$= \sum_{n=0}^{b-a-1} \left(\frac{1}{n+a} - \frac{1}{n+b} \right) + \sum_{n=b-a}^{N} \left(\frac{1}{n+a} - \frac{1}{n+b} \right)$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} - \sum_{n=0}^{b-a-1} \frac{1}{n+b} + \sum_{n=b-a}^{N} \frac{1}{n+a} - \sum_{n=b-a}^{N} \frac{1}{n+b}$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} - \sum_{n=b-a}^{2b-2a-1} \frac{1}{n+a} + \sum_{n=b-a}^{N} \frac{1}{n+a} - \sum_{n=b-a}^{N} \frac{1}{n+b}$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} + \sum_{n=2b-2a}^{N} \frac{1}{n+a} - \sum_{n=b-a}^{N} \frac{1}{n+b}$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} + \sum_{n=2b-2a}^{N} \frac{1}{n+a} - \sum_{n=2b-2a}^{N+b-a} \frac{1}{n+a}$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} - \sum_{n=N+1}^{N+b-a} \frac{1}{n+a}$$

$$= \sum_{n=0}^{b-a-1} \frac{1}{n+a} - \sum_{n=N+a-b+1}^{N} \frac{1}{n+b} .$$

(This may also be checked by induction on N.) Now the second summation is the sum of b–a terms, so clearly approaches 0 as $N \to \infty$. Hence

$$\sum^{\infty} \frac{b-a}{(n+a)(n+b)} = \sum^{b-a-1} \frac{1}{n+a}$$

$$n=0 \qquad\qquad n=0$$

$$= \frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{b-1}$$

$$= H_{b-1} - H_{a-1},$$

so dividing by (the non-zero) b–a yields a.

Now define

$$p(x) = \prod_{i=1}^{b-a} (x+a+i)$$

$$= (x+a+1)\cdot(x+a+2)\cdot \dots \cdot(x+b),$$

so that p(x) is an integer polynomial of degree b–a. Then, by the product rule,

$$p'(x) = \sum_{i=1}^{b-a} \prod_{[(1 \le j \le b-a) \,||\, (j \ne i)]} (x+a+j).$$

But since

$$\sum_{n=0}^{N} \frac{b-a}{(n+a)(n+b)} = \sum_{n=0}^{b-a+1} \frac{1}{n+a} - \sum_{n=N+a-b+1}^{N} \frac{1}{n+b}$$

$$= (b-a)S(a,b) - \sum_{n=N+a-b+1}^{N} \frac{1}{n+b},$$

and

$$\sum_{n=N+a-b+1}^{N} \frac{1}{n+b} = \frac{1}{N+a+1} + \frac{1}{N+a+2} + \dots + \frac{1}{N+b}$$

$$= \frac{p'(N)}{p(N)},$$

we have also the result asserted in b.

Here is an example. Consider a=2 and b=6. Then

$$\begin{aligned}
S(a,b) &= S(2,6) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+6)} \\
&= \frac{1}{6-2}\left(H_5 - H_1 \right) \\
&= \frac{1}{4}\left[(1+1/2+1/3+1/4+1/5)-1 \right] \\
&= \frac{1}{4} \cdot \frac{77}{60}, \quad \text{and} \\
p(x) &= (x+3)(x+4)(x+5)(x+6) \\
&= x^4+18x^3+119x^2+342x+360, \quad \text{whence} \\
p'(x) &= 4x^3+54x^2+238x+342, \quad \text{so that}
\end{aligned}$$

$$0 \le N \Rightarrow \sum_{n=0}^{N} \frac{1}{(n+2)(n+6)} = \frac{1}{4}\left(\frac{77}{60} - \frac{4N^3+54N^2+238N+342}{N^4+18N^3+119N^2+342N+360} \right)$$

$$= \frac{77}{240} - \frac{4N^3+54N^2+238N+342}{4N^4+72N^3+476N^2+1368N+1440}.$$

Also solved by John M. Sayer.