Problem for 2001 October

Communicated by Dan Jurca

Compute

$$\int_0^\infty \frac{x^{50}}{x^{100}+1}\,dx.$$

More generally, discuss

$$\int_0^\infty \frac{x^{a}}{x^{b}+1}\,dx.$$

Solution by Dan Jurca (also solved by Rudy Horne)

We consider the second integral. One can show that the integral exists if and only if $b < 0$ and $b-1 < a$, or if $0 < b$ and $a < b-1$; moreover, evaluating the integral in the first case reduces to the second case. Therefore assume that $0 < b$. Let $t=1/(x^b+1)$. Then $x^b=1/t-1$, so that $bx^{b-1}dx=-1/t^2dt$, from which

$$dx = -\frac{1}{b}\cdot\frac{1}{t^2}\cdot\frac{(1-t)^{1/b-1}}{t^{1/b-1}}\,dt, \quad \text{whence}$$

$$\begin{aligned}
\int_0^\infty \frac{x^a}{x^b+1}\,dx &= \frac{1}{b}\int_0^1 \left(\frac{1}{t}-1\right)^{a/b}\cdot t\cdot\frac{1}{t^2}\cdot(1-t)^{1/b-1}\cdot\frac{1}{t^{1/b-1}}\,dt \\
&= \ldots \\
&= \frac{1}{b}\int_0^1 t^{-(a+1)/b}(1-t)^{(a-b+1)/b}dt \\
&= \frac{1}{b}\int_0^1 t^{(b-a-1)/b-1}(1-t)^{(a+1)/b-1}dt
\end{aligned}$$

$$= \frac{1}{b} B\left(\frac{b-a-1}{b}, \frac{a+1}{b} \right) \quad \text{if } 0 < \frac{a+1}{b} \text{ and } 0 < \frac{b-a-1}{b}$$

where B is the Beta function [1, p.258]

$$= \frac{1}{b} \frac{\Gamma((a+1)/b)\Gamma((b-a-1)/b)}{\Gamma(1)} \quad \text{by [1], 6.2.2}$$

$$= \frac{1}{b} \Gamma\left(\frac{a+1}{b} \right) \Gamma\left(\frac{b-a-1}{b} \right)$$

$$= \frac{1}{b} \Gamma\left(\frac{a+1}{b} \right) \Gamma\left(1- \frac{a+1}{b} \right)$$

$$= \frac{1}{b} \cdot \pi\csc\left(\pi \frac{a+1}{b} \right) \quad \text{by the reflection formula [1], p. 256}$$

$$= \frac{\pi}{b\sin\left((a+1)/b\cdot\pi \right)} .\backslash C$$

Hence, if $0 < b$ and $-1 < a < b-1$, then

$$\int_0^\infty \frac{x^a}{x^b+1} dx = \frac{\pi}{b\sin\left((a+1)/b\cdot\pi \right)} .$$

In particular

$$1 < a \Rightarrow \int_0^\infty \frac{x^a}{x^{2a}+1} dx = \frac{\pi}{2a\sin\left((a+1)/2a\cdot\pi \right)}$$

$$= \frac{\pi}{2a\cos(\pi/2-(a+1)/2a\cdot\pi)} = \frac{\pi}{2a} \cdot\sec \frac{\pi}{2a} \text{, so}$$

$$\int_0^\infty \frac{x^{50}}{x^{100}+1} dx = \frac{\pi}{100} \cdot\sec \frac{\pi}{100}$$

$$\approx 0.031431436052208065886515 77.$$

We may also observe that

$$\int_0^\infty \frac{x^a}{x^{2a}+1}\,dx \sim \frac{\pi}{2a}\text{; i.e.,}$$

$$\lim_{a\to\infty} \frac{\int_0^\infty \frac{x^a}{x^{2a}+1}\,dx}{\frac{\pi}{2a}} = 1.$$

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[1] Handbook of Mathematical Functions, edited by Milton Abramowitz and Irene Stegun