Problem for 2002 March

Proposed by Dan Jurca

For $0 \le n$ let H_n be the sum of the first n terms of the harmonic series; *i.e.*,

$$H_n = \begin{cases} 0 & \text{if } n=0; \\ \sum_{i=1}^{n} \frac{1}{i} & \text{if } 1 \le n. \end{cases}$$

Find a formula for

$$\sum_{i=0}^{n} iH_i.$$

The proposer encountered this sum when analyzing a certain variation of the quicksort algorithm.

Solution by the proposer

Proposition.

$$0 \le n \Rightarrow \sum_{i=0}^{n} iH_i = \frac{n^2+n}{2} H_n - \frac{n^2-n}{4}$$

Proof.

The formula obviously holds if n=0; now suppose that $1 \le n$ and

$$\begin{aligned}
\sum_{i=0}^{n-1} iH_i &= \frac{(n-1)^2+(n-1)}{2} H_{n-1} - \frac{(n-1)^2-(n-1)}{4} \\
&= \frac{n^2-2n+1+n-1}{2} H_{n-1} - \frac{n^2-2n+1-n+1}{4} \\
&= \frac{n^2-n}{2} \left(H_n - \frac{1}{n} \right) - \frac{n^2-3n+2}{4} \\
&= \frac{n^2-n}{2} H_n - \frac{n-1}{2} - \frac{n^2-3n+2}{4} \\
&= \frac{n^2-n}{2} H_n - \frac{n^2-3n+2+2n-2}{4} \\
&= \frac{n^2-n}{2} H_n - \frac{n^2-n}{4} \text{; then}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{n} iH_i &= \frac{n^2-n}{2} H_n + nH_n - \frac{n^2-n}{4} \\
&= \frac{n^2+n}{2} H_n - \frac{n^2-n}{4} ,
\end{aligned}$$

so the proposition follows by induction on n.