# Problem for 2002 April

Proposed by Dan Jurca

Find a formula for the following sum, where n is a nonnegative integer.

$$\sum_{1 \le i < j \le n} ij$$

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Solution by the proposer

We show that the sum equals $[(n(3n+2)(n^2-1))/24]$. For with

$$S_n = \sum_{1 \le i < j \le n} ij, \quad \text{we have}$$

$$S_n = \begin{cases} 0 & \text{if } n=0, \\ \displaystyle\sum_{1 \le i < j \le n-1} ij + \left( \sum_{i=1}^{n-1} i \right) \times n & \text{if } 1 \le n; \text{ so that} \end{cases}$$

$$S_n = \begin{cases} 0 & \text{if } n=0, \\ S_{n-1} + \dfrac{(n-1)n^2}{2} & \text{if } 1 \le n. \end{cases}$$

Now the asserted formula obviously holds if n=0; suppose that $1 \le n$ and

$$S_{n-1} = \frac{(n-1)[3(n-1)+2][(n-1)^2-1]}{24}; \quad \text{then}$$

$$\begin{aligned}
S_n &= S_{n-1} + \frac{(n-1)n^2}{2} \\
&= \frac{(n-1)(3n-1)(n^2-2n)+12(n-1)n^2}{24} \\
&= \frac{(n-1)n[(3n-1)(n-2)+12n]}{24} \\
&= \frac{(n-1)n[3n^2-7n+2+12n]}{24} \\
&= \frac{(n-1)n(3n^2+5n+2)}{24} \\
&= \frac{(n-1)n(n+1)(3n+2)}{24} \\
&= \frac{n(3n+2)(n^2-1)}{24},
\end{aligned}$$

so that by induction the formula holds for each nonnegative integer n.