Problem for 2002 December

Communicated by Dan Jurca

The following problem appears on page 59 of *A Course of Modern Analysis* by E.T. Whittaker and G.N. Watson.

Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

is equal to [$1/((1-z)^2)$] when $|z| < 1$ and is equal to [$1/(z(1-z)^2)$] when $1 < |z|$.

Solution by Dan Jurca

First suppose $|z| \neq 1$. We shall show

$$1 \le N \Rightarrow \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^N}{(1-z)^2(1-z^{N+1})}.$$

(We take $0^0=1$ here.) First suppose N=1. Then

$$\sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{z^{1-1}}{(1-z^1)(1-z^{1+1})}$$

$$= \frac{1}{(1-z)(1-z^2)}$$

$$= \frac{1-z}{(1-z)^2(1-z^2)},$$

so the assertion certainly holds if N=1. Next suppose $2 \le N$ and

$$\sum_{n=1}^{N-1} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^{N-1}}{(1-z)^2(1-z^N)}. \quad \text{Then}$$

$$\sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1-z^{N-1}}{(1-z)^2(1-z^N)} + \frac{z^{N-1}}{(1-z^N)(1-z^{N+1})}$$

$$= \frac{(1-z^{N-1})(1-z^{N+1})+(1-z)^2z^{N-1}}{(1-z)^2(1-z^N)(1-z^{N+1})}$$

$$= \frac{1-z^{N-1}-z^{N+1}+z^{2N}+z^{N-1}-2z^N+z^{N+1}}{(1-z)^2(1-z^N)(1-z^{N+1})}$$

$$= \frac{(1-z^N)^2}{(1-z)^2(1-z^N)(1-z^{N+1})}$$

$$= \frac{1-z^N}{(1-z)^2(1-z^{N+1})},$$

so that the assertion holds for each N, $1 \le N$, by induction on N. Therefore, if $|z| < 1$, then $z^N \to 0$ as $N \to \infty$, whence

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

$$= \frac{1-z^N}{}$$

$$\lim_{N\to\infty} (1-z)^2(1-z^{N+1})$$

$$= \frac{1}{(1-z)^2}.$$

Next suppose 1 < |z|. Then with w=1/z we have |w| < 1, so that

$$\sum_{n=1}^{\infty} \frac{w^{n-1}}{(1-w^n)(1-w^{n+1})} = \frac{1}{(1-w)^2}.$$

Thus, replacing w with 1/z and negating each factor in the denominators, as we may,

$$\sum_{n=1}^{\infty} \frac{1/z^{n-1}}{(1/z^n-1)(1/z^{n+1}-1)} = \frac{1}{(1/z-1)^2}, \quad \text{so}$$

$$\sum_{n=1}^{\infty} \frac{z^n z^{n+1}/z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so}$$

$$\sum_{n=1}^{\infty} \frac{z^{n+2}}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so}$$

$$z^3 \sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{z^2}{(1-z)^2}, \quad \text{so that, finally,}$$

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z(1-z)^2},$$

as desired.

Also solved by Farid El-Mouchrif and John M. Sayer