Problem for 2003 February

Communicated by Dan Jurca

Let

$$
\begin{aligned}
N &= 4444^{4444},\\
A &= \text{the sum of the (decimal) digits of } N \text{, and}\\
B &= \text{the sum of the (decimal) digits of } A.
\end{aligned}
$$

What is the sum of the (decimal) digits of B?

Solution by anonymous solver

For each natural number $M=\sum_{i=0}^{n} d_i 10^i$ $(0 \le d_i < 10)$ we have $M \equiv \sum_{i=0}^{n} d_i \pmod 9$ since $10^i \equiv 1 \pmod 9$ $\forall i \in \mathbf{N}$. Let $N=4444^{4444}$, so if A= sum of digits of N, B= sum of digits of A, and C= sum of digits of B, then $N \equiv A \equiv B \equiv C \pmod 9$. Using Euler's φ-function theorem we have $N \equiv 7^{4444} \equiv 7^4 \equiv 7 \pmod 9$ since $4444 \equiv 7 \pmod 9$, $\varphi(9)=6$, and $4444 \equiv 4 \pmod 6$. Now N has $1+\lfloor \log_{10} N \rfloor = 16{,}211$ digits, and so $A \le 16{,}211\times 9 = 145{,}899$, a 6-digit number, and therefore $B \le 6\times 9 = 54$, and consequently $B \in \{7,16,25,34,43,52\}$. Thus, the sum of the digits of B is 7.

Remark by Dan Jurca:

Computing N one finds that A=72,601, B=16, so that, finally, C=7, as shown above.

These computations were performed on a desktop PC with a 200 MHz Pentium Pro processor, and the times were as follows: 0.215512 seconds to compute N in base 2^{32}, 0.519401 seconds to convert to decimal, and 0.000719 seconds to add the digits, obtaining A.

Solution also by James Dalton, Kurt Luoto, Farid El Mouchrif, Rosta Farzan, Gagan Sekhon, Michael Thompson