Problem for 2003 October

Proposed by Dan Jurca

For subsets A and B of some (universal) set S the *symmetric difference* A\triangle B is defined as follows.

$$\begin{aligned} A \triangle B &= (A \cup B) - (A \cap B) \\ &= \{x \in A \cup B \mid x \notin A \cap B\} \end{aligned}$$

It is well-known that this is an associative operation on S; *i.e.*, for three subsets A, B, and C of S

$$A \triangle (B \triangle C) = (A \triangle B) \triangle C.$$

Find a proof of this associative property which is not too tedious to read.

Solution by the proposer

We observe that A\triangle B=(A∪B)∩(A∩B)′, where ′ is set complement, recall the following notation

$$\begin{aligned} \mathcal{P}(S) &= \{A \mid A \subset S\}, \quad \text{the set of subsets of } S \\ 2^S &= \{f: S \to \{0,1\} \mid f \text{ is a function}\}, \quad \text{the set of functions from } S \text{ to } \{0,1\} \end{aligned}$$

$$A \subset S \quad \Rightarrow \chi_A : S \to \{0,1\} \text{ by } \chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}, \text{ the characteristic function of } A,$$

and observe that $\chi : \mathcal{P}(S) \to 2^S$ by $\chi(A) = \chi_A$ is a bijection. Also one has at once that

$$\begin{aligned}
\chi_{A'} &= 1-\chi_A \\
\chi_{A\cup B} &= \chi_A+\chi_B-\chi_A\chi_B \\
\chi_{A\cap B} &= \chi_A\chi_B \quad \text{so that} \\
\chi_{A\triangle B} &= \chi_{(A\cup B)\cap(A\cap B)'} \\
&= \chi_{A\cup B}\chi_{(A\cap B)'} \\
&= \chi_{A\cup B}(1-\chi_{A\cap B}) \\
&= \chi_{A\cup B}-\chi_{A\cup B}\chi_{A\cap B} \\
&= \chi_A+\chi_B-\chi_A\chi_B-(\chi_A+\chi_B-\chi_A\chi_B)\chi_A\chi_B \\
&= \chi_A+\chi_B-2\cdot\chi_A\chi_B.
\end{aligned}$$

Therefore

$$\begin{aligned}
\chi_{A\triangle(B\triangle C)} &= \chi_A+\chi_{B\triangle C}-2\cdot\chi_A\chi_{B\triangle C} \\
&= \chi_A+\chi_B+\chi_C-2\cdot\chi_B\chi_C-2\cdot\chi_A(\chi_B+\chi_C-2\cdot\chi_B\chi_C) \\
&= \chi_A+\chi_B+\chi_C-2\cdot\chi_B\chi_C-2\cdot\chi_A\chi_B-2\cdot\chi_A\chi_C+4\cdot\chi_A\chi_B\chi_C \\
&= \chi_A+\chi_B-2\cdot\chi_A\chi_B+\chi_C-2\cdot\chi_A\chi_C-2\cdot\chi_B\chi_C+4\cdot\chi_A\chi_B\chi_C \\
&= \chi_A+\chi_B-2\cdot\chi_A\chi_B+\chi_C-2(\chi_A+\chi_B-2\cdot\chi_A\chi_B)\chi_C \\
&= \chi_{A\triangle B}+\chi_C-2\cdot\chi_{A\triangle B}\chi_C \\
&= \chi_{(A\triangle B)\triangle C},
\end{aligned}$$

so that $\chi(A\triangle(B\triangle C))=\chi((A\triangle B)\triangle C)$, and since χ is one-to-one it follows that $A\triangle(B\triangle C)=(A\triangle B)\triangle C$, proving associativity of $\triangle$.

Also solved by Kirk Demlinger, Udayabaskar Nachimuthu, Bill Nico, and John Sayer