Problem for 2004 February

Proposed by Dan Jurca

Problem number 174 (from Murray Klamkin) in the book *Which Way did the Bicycle Go?* by Joseph D. E. Konhauser, Dan Velleman, and Stan Wagon appears as follows.

Find, with an error of no more than 5%, the numerical value of

$$\int_1^{100} x^x \, dx.$$

The answer $1.80086 \cdot 10^{199}$ is given on page 211.

Find a more accurate approximation.
Precisely, find an approximation A of the above integral I with a relative error less than 10^{-6}; *i.e.*, we require

$$\frac{|I-A|}{I} < 10^{-6}.$$

Solution by the proposer

We begin by observing $(x^x)' = x^x(1+\ln x)$, and next by l'Hôpital's rule and the fundamental theorem of calculus

$$\lim_{x\to\infty} \frac{\int_1^x t^t \, dt}{x^x/(1+\ln x)} = \lim_{x\to\infty} \frac{(1+\ln x)\int_1^x t^t \, dt}{x^x}$$

$$= \lim \quad \frac{1}{} \int^x t^t \, dt + x^x(1+\ln x)$$

$$\underset{x\to\infty}{} \quad \frac{x \int_1}{x^x(1+\ln x)}$$

$$=1+\lim_{x\to\infty}\frac{\int_1^x t^t\,dt}{x\cdot x^x(1+\ln x)}$$

$$=1+\lim_{x\to\infty}\frac{x^x}{x^x(1+\ln x)+x[x^x(1+\ln x)^2+x^{x-1}]}$$

$$=1+0$$

$$=1.$$

One writes

$$\int_1^x t^t\,dt \sim \frac{x^x}{1+\ln x},$$

and this suggests that for large x one may approximate

$$\int_1^x t^t\,dt$$

with $x^x/(1+\ln x)$; for example

$$\int_1^{100} t^t\,dt \approx \frac{100^{100}}{1+\ln 100} = \frac{10^{200}}{1+\ln 100} \approx 1.7841\times10^{199}.$$

This approximation, however motivated, provides no information about the error of the approximation; hence we argue as follows.

For each a, $1 < a < 100$, we write

$$i_a = \int^{a} x^x\,dx \quad\text{and}\quad I_a = \int^{100} x^x\,dx.$$

$$\int_{1} \qquad\qquad \int_{a}$$

Then obviously

$$\begin{aligned} I &= \int_{1}^{100} x^x\,dx \\ &= \int_{1}^{a} x^x\,dx + \int_{a}^{100} x^x\,dx \\ &= i_a + I_a, \quad \text{and} \\ i_a &< (a-1)\cdot a^a. \end{aligned}$$

(We shall later choose $a=e^{4.5} \approx 90.02$, and then $i_a < 7.5\times10^{177}$. Hence it will remain to approximate I_a.)

Next with $u_1=1/(1+\ln x)$ and $dv=x^x(1+\ln x)\,dx$ we have

$$du_1 = -\frac{1}{x(1+\ln x)^2}\,dx \quad \text{and} \quad v=x^x,$$

so using integration by parts

$$I_a = x^x\cdot\frac{1}{1+\ln x}\Bigg|_{a}^{100} + \int_{a}^{100} \frac{1}{x(1+\ln x)^2}\cdot x^x\,dx;$$

now with $u_2=1/[x(1+\ln x)^3]$ and dv as before we have

$$du_2 = -\frac{4+\ln x}{x^2(1+\ln x)^4}\,dx \quad \text{and} \quad v \text{ as before},$$

so again using integration by parts

$$I_a = x^x\left(\frac{1}{1+\ln x} + \frac{1}{x(1+\ln x)^3}\right)\Bigg|^{100} + \int^{100} \frac{4+\ln x}{x^2(1+\ln x)^4}\cdot x^x\,dx;$$

a a

finally, with $u_3=(4+\ln x)/[x^2(1+\ln x)^5]$ and dv as before,

$$du_3=-\frac{27+14\ln x+2(\ln x)^2}{x^3(1+\ln x)^6}\,dx \quad \text{and} \quad v \text{ as before,}$$

so once again using integration by parts

$$I_a=x^x\left(\frac{1}{1+\ln x}+\frac{1}{x(1+\ln x)^3}+\frac{4+\ln x}{x^2(1+\ln x)^5}\right)\Bigg|_a^{100}+\int_a^{100}\frac{27+14\ln x+2(\ln x)^2}{x^3(1+\ln x)^6}\cdot x^x\,dx.$$

Now with $\varphi:[a,100]\to\mathbf{R}$ by

$$\varphi(x)=\frac{27+14\ln x+2(\ln x)^2}{x^3(1+\ln x)^6}$$

$$\text{we find}\quad \varphi'(x)=-\frac{229+189\ln x+56(\ln x)^2+6(\ln x)^3}{x^4(1+\ln x)^7}$$

so that $\varphi' < 0$ whence φ\searrow, and $\varphi_{max}=\varphi(a)$; therefore

$$\int_a^{100}\varphi(x)\cdot x^x\,dx < \varphi(a)\int_a^{100}x^x\,dx=\varphi(a)I_a.$$

Now let

$$A_a=x^x\left[\frac{1}{1+\ln x}+\frac{1}{x(1+\ln x)^3}+\frac{4+\ln x}{x^2(1+\ln x)^5}\right]_a^{100};$$

then we have $A_a < I_a < A_a+\varphi(a)I_a$, and we shall show that $A_{e^{4.5}}$ is a sufficiently accurate approximation of I.

Now clearly $A_a < i_a+A_a < i_a+I_a=I$, so that $0 < I-A_a$ and $|I-A_a|=I-A_a$.

Next, $I=i_a+I_a < i_a+A_a+\varphi(a)I_a$, so $I-A_a < i_a+\varphi(a)I_a$. It therefore follows that

$$\begin{aligned}
\frac{|I-A_a|}{I} &= \frac{I-A_a}{I} \\
&< \frac{i_a+\varphi(a)I_a}{I} \\
&= \frac{i_a}{I} + \varphi(a)\frac{I_a}{I} \\
&< \frac{i_a}{A_a} + \varphi(a),
\end{aligned}$$

since $A_a < I$ and $I_a < I$.

Now with $a=e^{4.5}$ we recall $i_a < 7.5\times10^{177}$ and find (using a HP-42S calculator) that

$$\begin{aligned}
A_a &\approx 1.784636555\times10^{199} \\
\text{so that } \frac{i_a}{A_a} &< 5\times10^{-22} \\
\text{and} \varphi(a) &\approx 6.46336778\times10^{-9}, \\
\text{so that } \quad \frac{|I-A_a|}{I} &< 7\times10^{-9}.
\end{aligned}$$

Hence we have $I \approx 1.7846366\times10^{199}$ with a relative error less than 5×10^{-8}, so that each digit in this approximation is significant.