Problem for 2004 June

Communicated by Dan Jurca

According to the 2004 May issue of the Canadian Mathematics journal *Crux Mathematicorum* the following problem appeared in the National Round of the XXXVI Spanish Mathematical Olympiad.

Show that there exists no function $f:\mathbf{N}\to\mathbf{N}$ such that for each n

$$f(f(n))=n+1.$$

Solution by Dan Jurca

Suppose otherwise; *i.e.*, $f:\mathbf{N}\to\mathbf{N}$ and for each $n \in \mathbf{N}$: $f(f(n))=n+1$. Then

$$\begin{aligned} f(f(f(n))) &= f[f(f(n))]=f[n+1] \quad \text{and} \\ &= f(f[f(n)])=f(n)+1, \quad \text{so that} \end{aligned}$$

$$n \in \mathbf{N} \Rightarrow f(n+1)=f(n)+1.$$

It follows by an easy induction on k that $1 \le k \Rightarrow f(n+k)=f(n)+k$. Now suppose $f(1)=y$. Then

$$\begin{aligned} f(n) &= f(1+(n-1)) \\ &= f(1)+(n-1) \\ &= y+(n-1), \quad \text{so that} \\ f(n) &= y+n-1; \quad \text{but then} \\ f(y) &= 2y-1 \quad \text{and} \end{aligned}$$

$$\begin{aligned}
f(f(n)) &= f(y+n-1) \\
&= f(y)+n-1 \\
&= (2y-1)+n-1 \\
&= 2y+n-2, \quad \text{and also} \\
f(f(n)) &= n+1, \quad \text{whence} \\
2y+n-2 &= n+1, \quad \text{and} \\
2y &= 3,
\end{aligned}$$

which contradicts $y=f(1) \in \mathbf{N}$; therefore no such f exists.