Problem for 2004 September and October

Proposed by Dan Jurca

For each positive integer n define S(n) as follows.

$$S(n) = \sum_{b=2}^{n} \log_b n$$

$$=\log_2 n+\log_3 n+\log_4 n+\ldots+\log_n n$$

Determine whether

$$\lim_{n\to\infty} \frac{S(n)}{n}$$

exists; and if it exists, determine its value.

Solution by the proposer

We have

$$S(n)= \sum_{b=2}^{n} \log_b n= \sum_{b=2}^{n} \frac{\ln n}{\ln b} =\ln n \sum_{b=2}^{n} \frac{1}{\ln b}.$$

The function $(1,\infty)\to\mathbf{R}$ by $t\to 1/\ln t$ assumes only positive values and strictly decreases, so

$$1 < x \Rightarrow \frac{1}{\ln(x+1)} < \int^{x+1} \frac{dt}{\ln t} < \frac{1}{\ln x};$$

x

hence (by summing)

$$3 \le n \Rightarrow \int_2^n \frac{dt}{\ln t} + \int_n^{n+1} \frac{dt}{\ln t} < \sum_{b=2}^{n} \frac{1}{\ln b} = \frac{1}{\ln 2} + \sum_{b=3}^{n} \frac{1}{\ln b}$$

$$< \frac{1}{\ln 2} + \sum_{b=3}^{n} \int_{b-1}^{b} \frac{dt}{\ln t}$$

$$= \frac{1}{\ln 2} + \int_2^n \frac{dt}{\ln t}$$

whence

$$3 \le n \Rightarrow \frac{\ln n}{n} \cdot \int_2^n \frac{dt}{\ln t} < \frac{S(n)}{n} < \frac{\ln n}{n} \cdot \frac{1}{\ln 2} + \frac{\ln n}{n} \cdot \int_2^n \frac{dt}{\ln t} .$$

Now

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0 \quad \text{and} \quad \int_2^\infty \frac{dt}{\ln t} = \infty \quad \left(\text{since } 1 < t \Rightarrow \frac{1}{t} < \frac{1}{\ln t}\right)$$

so by l'Hôpital's rule and the fundamental theorem of calculus

$$\lim_{x\to\infty} \frac{\ln x}{x} \cdot \int_2^x \frac{dt}{\ln t} = 1;$$

therefore by the squeeze theorem

$$\lim_{n\to\infty} \frac{S(n)}{n} = 1.$$

Also solved by Dennis Eichhorn and Sarah Frey