Problem for 2005 February

Communicated by Dan Jurca

Show that there do not exist rational numbers x and y such that $x^2+y^2=11$.

Solution by Dan Jurca

We shall prove the more general result that if n is a positive integer and $n \equiv 3 \pmod 4$, then there do not exist rational numbers x and y such that $x^2+y^2=n$. Assuming otherwise leads to a contradiction as follows. For there are sixteen possibilities for the pair of rational numbers: even/even and even/even, even/even and even/odd, *etc*., and if we suppose that each of x and y has been reduced to lowest terms, then there are only nine such possibilities. The six cases x= even/odd and y= even/odd, x= even/odd and y= odd/even, x= odd/even and y= even/odd, x= odd/even and y= odd/odd, x= odd/odd and y= odd/even, and x= odd/odd and y= odd/odd are all eliminated condidering parity. (We cannot have an even number equal to an odd number.) Consider the case x= even/odd and y= odd/odd. Then, squaring, we have $\text{even}^2\ \text{odd}^2+\text{odd}^2\ \text{odd}^2=\text{odd}^2\ \text{odd}^2\times n$. But since each even number squared is an even number, and each odd number squared is odd, the left side here is congruent to 1 (mod 4), and the right side is congruent to 3. Of course the same holds for x= odd/odd and y= even/odd. Therefore there is but one remaining case: x= odd/even and y= odd/even. Thus there exist integers a,b,c,d,e,α, and γ such that $1 \le \alpha \le \gamma$ and

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\begin{aligned}
\left[\frac{2a+1}{2^{\alpha}(2b+1)}\right]^2+\left[\frac{2c+1}{2^{\gamma}(2d+1)}\right]^2 &= 4e+3, \quad \text{so}\\
2^{2\gamma}(2a+1)^2(2d+1)^2+2^{2\alpha}(2b+1)^2(2c+1)^2 &= 2^{2(\alpha+\gamma)}(2b+1)^2(2d+1)^2(4e+3), \quad \text{so}\\
2^{2(\gamma-\alpha)}(2a+1)^2(2d+1)^2+(2b+1)^2(2c+1)^2 &= 2^{2\gamma}(2b+1)^2(2d+1)^2(4e+3), \quad \text{whence}\\
\alpha &= \gamma, \quad \text{so}\\
(2a+1)^2(2d+1)^2+(2b+1)^2(2c+1)^2 &= 4^{\gamma}\times\text{an odd integer},
\end{aligned}
$$

but here the left side is congruent to 2 (mod 4) and the right side is congruent to 0 (mod 4).

Generalization also solved by Massoud Malek