Problem for 2005 August

Communicated by Dan Jurca

According to the *Hungarian Problem Book II* the following problem appeared as number 3 in the 1911 Eötvös competition.

Prove that 3^n+1 is not divisible by 2^n for any integer $n > 1$.

Solution by Dan Jurca

We shall show that if n is even, then 3^n+1 equals 2 times an odd integer; and if n is odd, then 3^n+1 equals 4 times an odd integer. Precisely, we define integer sequences $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ as follows:

$$\begin{aligned} a_0 &= 0; \\ 1 \le k \Rightarrow a_k &= 9a_{k-1}+2 \\ 0 \le k \Rightarrow 3^{2k}+1 &= 2(2a_k+1) \end{aligned}$$

$$\begin{aligned} b_0 &= 0; \\ 1 \le k \Rightarrow b_k &= 9b_{k-1}+3, \quad \text{and claim} \\ 0 \le k \Rightarrow 3^{2k+1}+1 &= 4(2b_k+1). \end{aligned}$$

For we have

$$\begin{aligned} 3^0+1 &= 1+1=2=2\times 1 = 2\times(2\cdot 0+1)=2(2a_0+1), \quad \text{and} \\ 3^1+1 &= 3+1=4=4\times 1 = 4\times(2\cdot 0+1)=4(2b_0+1). \end{aligned}$$

Next, if $1 \le k$, and

$$\begin{aligned} 3^{2k-2}+1 &= 2(2a_{k-1}+1), \quad \text{then} \\ 3^{2k-2} &= 4a_{k-1}+1, \quad \text{so} \end{aligned}$$

$$
\begin{aligned}
3^{2k} &= 36a_{k-1}+9, \quad \text{and} \\
3^{2k}+1 &= 36a_{k-1}+10 \\
&= 2[18a_{k-1}+5] \\
&= 2[2(9a_{k-1}+2)+1] \\
&= 2(2a_k+1) \\
3^{2k-1}+1 &= 4(2b_{k-1}+1), \quad \text{then} \\
3^{2k-1} &= 8b_{k-1}+3, \quad \text{so} \\
3^{2k+1} &= 72b_{k-1}+27, \quad \text{and} \\
3^{2k+1}+1 &= 72b_{k-1}+28 \\
&= 4(18b_{k-1}+7) \\
&= 4[2(9b_{k-1}+3)+1] \\
&= 4(2b_k+1),
\end{aligned}
$$

whence the claims follow by induction on k.

It follows at once that $2 \le n \Rightarrow 2^n$ does not divide 3^n+1.

Also solved by Kelly Hubble.