Problem for 2006 October

Communicated by Dan Jurca

One may compute the value of a commonly occurring function f of two positive integers by the following formula.

$$f(m,n)=m+n-mn+2\sum_{k=1}^{n-1}\left\lfloor \frac{km}{n} \right\rfloor$$

Identify this function, and prove the correctness of your result.

Solution by Dan Jurca

The following three propositions will show that f=gcd.

Proposition 1. If $x \in \mathbf{R}$, then $\lfloor x\rfloor+\lfloor -x\rfloor = 0$ if $x \in \mathbf{Z}$ and $\lfloor x\rfloor+\lfloor -x\rfloor = -1$ if $x \notin \mathbf{Z}$.
Proof.
Suppose $x \in \mathbf{R}$. Then there exists (a unique) θ such that $0 \le \theta < 1$ and $x=\lfloor x\rfloor+\theta$. Clearly $\theta = 0$ iff $x \in \mathbf{Z}$. Now

$$-x=-\lfloor x\rfloor-\theta = -\lfloor x\rfloor-1+(1-\theta)$$

and since $0 < 1-\theta \le 1$ while $1-\theta = 1$ iff $x \in \mathbf{Z}$, (else $0 < 1-\theta < 1$) we have

$$\lfloor -x\rfloor = \begin{cases} -\lfloor x\rfloor & \text{if } x \in \mathbf{Z} \\ -\lfloor x\rfloor-1 & \text{if } x \notin \mathbf{Z} \end{cases}$$

whence

$$\lfloor x\rfloor+\lfloor -x\rfloor = \begin{cases} 0 & \text{if } x \in \mathbf{Z} \\ -1 & \text{if } x \notin \mathbf{Z}. \end{cases}$$

Proposition 2. $f(m,n)=|\{k \in \mathbf{N} : 1 \le k \le n \text{ and } n|km\}|$.
Proof.

$$2\sum_{k=1}^{n-1} \left\lfloor \frac{km}{n} \right\rfloor = \sum_{k=1}^{n-1} \left(\left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor \frac{(n-k)m}{n} \right\rfloor \right).$$

Now $\left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor \frac{(n-k)m}{n} \right\rfloor = \left\lfloor \frac{km}{n} \right\rfloor + m + \left\lfloor \frac{-km}{n} \right\rfloor$,

and $\left\lfloor \frac{km}{n} \right\rfloor + \left\lfloor \frac{-km}{n} \right\rfloor = \begin{cases} 0 & \text{if } n|km \\ -1 & \text{if } n\not|km. \end{cases}$

Therefore with $g=g(m,n)=|\{k \in \mathbf{N} : 1 \le k \le n \text{ and } n|km\}|$, we have (since $n|nm$ but $n-1 < n$)

$$\begin{aligned} 2\sum_{k=1}^{n-1} \left\lfloor \frac{km}{n} \right\rfloor &= (g-1)m+(n-g)(m-1) \\ &= gm-m+mn-n-gm+g \\ &= -m-n+mn+g, \end{aligned}$$

so that $f(m,n)=g=g(m,n)$.

Proposition 3. $1 \le m$ and $1 \le n \Rightarrow g(m,n)=|\{k \in \mathbf{N} : 1 \le k \le n \text{ and } n|km\}|=\gcd(m,n)$.
Proof.
Let $k_0=\min\{k : 1 \le k \le n \text{ and } n|km\}$, so $1 \le k_0 \le n$, and $n|k_0m$. Clearly, since $n|k_0m$, then $1 \le q \Rightarrow n|(qk_0)m$. Moreover, if $n|km$, then $\exists q$ such that $k=qk_0$. For say $n|km$ and $k=k_0q+r$, where $0 \le r < k_0$. Then since $n|km$ we have $n|(k_0q+r)m$, so, since $n|k_0qm$, it follows that $n|rm$ as well; then since $r < k_0$, it follows that $r=0$. Thus $n|km$ iff k equals some multiple of k_0. Therefore, since $n|nm$, we have $k_0|n$, and $g(m,n)=n/k_0$. Therefore $g(m,n)|n$. Obviously $(n/k_0)|m$, since $n|(k_0m)$. It

follows that $g(m,n)|m$ as well. Now since $g(m,n)|m$ and $g(m,n)|n$, we have $g(m,n)|\gcd(m,n)$, so $g(m,n) \le \gcd(m,n)$.

Now suppose $n/\gcd(m,n)=k_1$, so $1 \le k_1 \le n$. Then

$$\frac{k_1 m}{n} = \frac{m}{n/k_1} = \frac{m}{\gcd(m,n)} \in \mathbf{N}$$

so that k_1 equals some multiple of k_0. Hence $k_0 \le k_1$, whence $\gcd(m,n)=n/k_1 \le n/k_0=g(m,n)$, so that $\gcd(m,n) \le g(m,n)$.

Thus $g(m,n)=\gcd(m,n)$.

By propositions 1, 2, and 3 it follows that $f(m,n)=\gcd(m,n)$; *i.e.*, f=gcd.

Also solved by Massoud Malek