Problem for 2007 January

Communicated by Dan Jurca

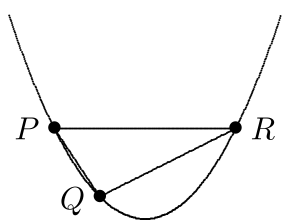
The following problem from the Final Round, Category C, of the 2002 Belarus Mathematical Olympiad appears in the 2006 November issue of the Canadian mathematics journal *Crux Mathematicorum.*

(*I. Zhuk*) Prove that a right-angled triangle can be inscribed in the parabola $y = x^2$ so that its hypotenuse is parallel to the x-axis if and only if the altitude from the right angle is equal to 1. (A triangle is inscribed in a parabola if all three vertices of the triangle are on the parabola.)

Solution by Dan Jurca

Consider the following diagram in which

a. the curve is the graph of the function $f(x) = x^2$,
b. the (rectangular) coordinates of the point P are $(-a, a^2)$,
c. the (rectangular) coordinates of the point Q are (b, b^2),
d. the (rectangular) coordinates of the point R are (a, a^2).



Then we have the following distances.

$$\begin{aligned} d(P,Q)^2 &= (-a-b)^2 + (a^2-b^2)^2 \\ d(Q,R)^2 &= (a-b)^2 + (a^2-b^2)^2 \\ d(P,R)^2 &= 4a^2 \end{aligned}$$

Triangle PQR is a right-angled triangle if and only if

- the three vertices P, Q, and R are distinct; *i.e.*, $a \neq \pm b$, and
- $d(P,Q)^2 + d(Q,R)^2 = d(P,R)^2$,

that is if and only if $a \neq \pm b$ and $(-a-b)^2 + (a^2-b^2)^2 + (a-b)^2 + (a^2-b^2)^2 = 4a^2$. This equality is equivalent to $(a^2-b^2)^2 = a^2 - b^2$, so that either $a^2 = b^2$, in which case triangle PQR is degenerate; or else $a^2 - b^2 = 1$, in which case the altitude from the vertex at the right angle Q is equal to 1.