Problem for 2007 April

Proposed by Dan Jurca

a.

Suppose

$$f:[0,1]\to\mathbf{R} \text{ by } f(x)=\begin{cases} 0 & \text{if } x=0 \\ x\sin\frac{1}{x} & \text{if } 0 < x \le 1. \end{cases}$$

Determine whether the length of the graph of f is finite or infinite.

b.

Suppose

$$g:[0,1]\to\mathbf{R} \text{ by } g(x)=\begin{cases} 0 & \text{if } x=0 \\ x^2\sin\frac{1}{x} & \text{if } 0 < x \le 1. \end{cases}$$

Determine whether the length of the graph of g is finite or infinite.

Solution by the proposer

a.

The graph of f is of infinite length. For $i=0,1,\dots$ let $x_i=2/((2i+1)\pi)$ and $y_i=f(x_i)=(-1)^i x_i$. The graph of f is a curve in the x-y plane which contains the points (x_i,y_i), $i=0,1,\dots$; it ``passes through'' (x_0,y_0), then (x_1,y_1), then (x_2,y_2), *etc*. The length of arc of the portion from (x_{i-1},y_{i-1}) to (x_i,y_i) clearly exceeds $|y_{i-1}|+|y_i|$, $i=1,2,\dots$. Thus

$$0 \le n \Rightarrow \int_{x_{n+1}}^{1} \sqrt{1+(f')^2} > |y_0|+|y_{n+1}|+2\sum_{i=1}^{n} |y_i|$$

$$= 2 + \quad 2 \quad + 4 \quad n \quad 1$$

$$\frac{\;}{\pi} \quad \frac{\;}{(2n+3)\pi} \quad \frac{\;}{\pi}\sum_{i=1} \frac{\;}{2i+1}$$

and since the series $1/3+1/5+1/7+\ldots$ diverges to ∞,

$$\int_0^1 \sqrt{1+(f')^2} = \lim_{a\to 0^+} \int_a^1 \sqrt{1+(f')^2} = \infty,$$

and the graph of f is of infinite length.

b.

We have

$$g'(x) = \begin{cases} 0 & \text{if } x=0, \\ 2x\sin(1/x)-\cos(1/x) & \text{if } 0 < x \le 1, \end{cases} \quad \text{so that}$$

$$0 < x \le 1 \Rightarrow \sqrt{1+(g'(x))^2} = \sqrt{1+4x^2\sin^2(1/x)-4x\sin(1/x)\cos(1/x)+\cos^2(1/x)}\,, \quad \text{and}$$

$$0 \le x \le 1 \Rightarrow \sqrt{1+(g'(x))^2} \le \sqrt{2+4x^2+4x}\,, \quad \text{whence}$$

$$\int_0^1 \sqrt{1+(g')^2} \le \int_0^1 \sqrt{4x^2+4x+2}\; dx$$

$$= \frac{1}{4}\left(-\sqrt{2}+3\sqrt{10} - \operatorname{arcsinh}(1)+\operatorname{arcsinh}(3) \right)$$

$$= \frac{1}{4}\left(3\sqrt{10} - \sqrt{2}+\ln \frac{3+\sqrt{10}}{1+\sqrt{2}} \right)$$

$$\approx 2.2524230726$$

so the graph of g is of finite length.

Remark. Numerical integration suggests that the length of the graph of g is approximately 1.9288.

Also solved by Massoud Malek and Grant Morgan
