Problem for 2007 July

Communicated by Dan Jurca

By the *repunit* R_n one means the integer written in base 10 as 111…111, the string of n 1's; *i.e.*,

$$R_n = \frac{10^n - 1}{9}.$$

Let S={p | p is a prime number and ∃n such that p divides R_n}; determine the set S.

Solution communicated by Dan Jurca

Obviously $2 \notin S$ and $5 \notin S$. We show S consists of all other primes. For suppose that p is a prime, $p \neq 2$, and $p \neq 5$. For each positive integer k we clearly have $R_k \equiv r \pmod{p}$ for some $r \in \{0,1,2,\dots,p-1\}$, and since this set is finite there exist positive integers m and n such that $m < n$ and $R_n \equiv R_m \pmod{p}$. Therefore $p \mid (R_n - R_m)$. But

$$\begin{aligned} R_n - R_m &= \frac{10^n-1}{9} - \frac{10^m-1}{9} \\ &= \frac{10^n-1-10^m+1}{9} \\ &= \frac{10^n-10^m}{9} \\ &= \frac{10^m(10^{n-m}-1)}{9} \\ &= 10^m \times \frac{10^{n-m}-1}{} \end{aligned}$$

9

$$=10^m \times R_{n-m}$$

and since p does not divide 10^m, we have $p|R_{n-m}$, whence $p \in S$.

Also solved by Eric Bahr, Massoud Malek, and Steven Schluchter