Problem for 2007 August

Communicated by Dan Jurca

The following problem appears on page 158 of *The IMO Compendium* by Dusan Djuki\'c, Vladimir Jankovi\'c, Ivan Mati\'c, and Nikola Petrovi\'c.

Consider the set $\mathbf{Q}^2$ of points in $\mathbf{R}^2$, both of whose coordinates are rational.

a. Prove that the union of segments with vertices from $\mathbf{Q}^2$ is the entire set $\mathbf{R}^2$.

b. Is the convex hull of $\mathbf{Q}^2$ (*i.e*, the smallest convex set in $\mathbf{R}^2$ that contains $\mathbf{Q}^2$) equal to $\mathbf{R}^2$?

i. Show that the assertion of a. is incorrect by exhibiting a point $(x,y) \in \mathbf{R}^2$ which is not contained in the union of segments each endpoint of which is a point in $\mathbf{Q}^2$.

ii. Show that if $\Im=\{\alpha \in \mathbf{R} \mid \alpha \notin \mathbf{Q}\}$ then the union of segments with endpoints in $\Im^2$ equals $\mathbf{R}^2$.

Solution by Dan Jurca

i. We show that the point $(\sqrt{2},\sqrt{3}) \in \mathbf{R}^2$ is not in a segment each endpoint of which lies in $\mathbf{Q}^2$. For suppose $(a,b) \in \mathbf{Q}^2$, $(c,d) \in \mathbf{Q}^2$, and that $(\sqrt{2},\sqrt{3}) \in l$, where l is the segment from (a,b) to (c,d). Then clearly $a \neq c$, else $\sqrt{2} = a \in \mathbf{Q}$; similarly $b \neq d$, else $\sqrt{3} = b \in \mathbf{Q}$. Hence with $m=(b-d)/(a-c)$, l is included in the line with equation $y-b=m(x-a)$, so that $\sqrt{3} = m\sqrt{2}-(ma-b)$. Now if $ma-b=0$, then $\sqrt{3} = m\sqrt{2}$; hence $\sqrt{3}/\sqrt{2} = m=\sqrt{6}/2$, from which $\sqrt{6} = 2m \in \mathbf{Q}$, which is false, since $\sqrt{6}$ is irrational. But if $(ma-b) \neq 0$, then by squaring we find

$$3=2m^2+(ma-b)^2-2m(ma-b)\sqrt{2},$$

and this yields (since $m \in \mathbf{Q}$ and $m \neq 0$)

$$\sqrt{2} = \ 3-2m^2-(ma-b)^2 \in \mathbf{Q},$$

$$-2m(ma-b)$$

another contradiction.

ii. We shall show that in fact each point $(x,y) \in \mathbf{R}^2$ is the midpoint of a segment each endpoint of which lies in $\Im^2$. There are four possibilities, as follows.

1. $x \in \Im$, $y \in \Im$. Consider the segment from $(x-1,y)$ to $(x+1,y)$.
2. $x \in \Im$, $y \in \mathbf{Q}$. Consider the segment from $(x,y-\sqrt{2})$ to $(x,y+\sqrt{2})$.
3. $x \in \mathbf{Q}$, $y \in \Im$. Consider the segment from $(x-\sqrt{2},y)$ to $(x+\sqrt{2},y)$.
4. $x \in \mathbf{Q}$, $y \in \mathbf{Q}$. Consider the segment from $(x-\sqrt{2},y-\sqrt{2})$ to $(x+\sqrt{2},y+\sqrt{2})$.