Problem for 2008 July

Proposed by Dan Jurca

Prove that for integers m and n with $1 \le m < n$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = 0.$$

Solution by the proposer

Let

$$S(m,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m;$$

we first show that

$$2 \le m \text{ and } 2 \le n \Rightarrow S(m,n) = n[S(m-1,n) - S(m-1,n-1)].$$

For

$$S(m-1,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^{m-1}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (-1)^i \frac{n}{i} \binom{n}{i} i^m \quad \text{and}$$

$$S(m-1,n-1) = \sum^{n-1} (-1)^i \binom{n-1}{i} i^{m-1}$$

$$i=0$$

$$= \frac{1}{n} \sum_{i=1}^{n-1} (-1)^i \frac{n-i}{i} \binom{n}{i} i^m$$

$$= \frac{1}{n} \sum_{i=1}^{n} (-1)^i \frac{n-i}{i} \binom{n}{i} i^m, \quad \text{whence}$$

$$S(m-1,n)-S(m-1,n-1) = \frac{1}{n} \sum_{i=1}^{n} (-1)^i \left(\frac{n}{i} - \frac{n-i}{i} \right) \binom{n}{i} i^m$$

$$= \frac{1}{n} \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^m$$

$$= \frac{1}{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m$$

$$= \frac{1}{n} S(m,n),$$

from which the result follows.

Next we claim $1 < n \Rightarrow S(1,n)=0$. For $1 < n \Rightarrow$

$$S(1,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} i$$

$$= \sum_{i=1}^{n} (-1)^i i \binom{n}{i}$$

$$= n \quad n \quad (-1)^i \binom{n-1}{i-1}$$

$$\sum_{i=1}$$

$$= -n \sum_{i=1}^{n} \binom{n-1}{i-1} 1^{(n-1)-(i-1)}(-1)^{i-1}$$

$$= -n \sum_{i=0}^{n-1} \binom{n-1}{i} 1^{(n-1)-i}(-1)^{i}$$

$$= -n(1-1)^{n-1}$$

$$=0.$$

We now have

$$2 < n \Rightarrow S(2,n) \quad = n[S(1,n) - S(1,n-1)] = 0, \quad (\text{since } 1 < n-1)$$

$$3 < n \Rightarrow S(3,n) \quad = n[S(2,n) - S(2,n-1)] = 0, \quad (\text{since } 2 < n-1)$$

etc., so that $1 \leq m < n \Rightarrow S(m,n)=0$.

Remark. In fact $(-1)^n S(m,n)$ equals the number of surjections from an m-element set to an n-element set; obviously this number equals zero if $m < n$.

Also solved by Bojan Basic (Serbia), Minghua Lin (China), and John M. Sayer

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