Problem for 2008 September

Proposed by Dan Jurca

Recall that the Fibonacci sequence $(F_n)_{n=0}^{\infty}=(0,1,1,2,3,5,8,13,21,34,\ldots)$ can be defined recursively as follows.

$$F_0=0,\quad F_1=1,\quad 2\le n \Rightarrow F_n=F_{n-1}+F_{n-2}$$

Find in closed form the solution of the following recurrence.

$$x_0=0\quad x_1=1,\quad 2\le n \Rightarrow x_n=1\cdot x_{n-1}+2\cdot x_{n-2}+3$$

Solution by the proposer

With $x_{-1}=-1$ and

$$A=\begin{pmatrix}1&2&1\\1&0&0\\0&0&1\end{pmatrix},\quad \mathbf{v}_n=\begin{pmatrix}x_n\\x_{n-1}\\3\end{pmatrix},\quad \text{so } \mathbf{v}_0=\begin{pmatrix}x_0\\x_{-1}\\3\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}$$

we have $0\le n \Rightarrow \mathbf{v}_n=A^n\mathbf{v}_0$. Now the characteristic polynomial of A, $\det(A-\lambda I_3)=-\lambda^3+2\lambda^2+\lambda-2$, so the eigenvalues of A are 2, 1, and -1, with associated eigenvectors

$$\begin{pmatrix}2\\1\\0\end{pmatrix},\quad \begin{pmatrix}-1\\-1\\2\end{pmatrix},\quad \text{and}\quad \begin{pmatrix}1\\-1\\0\end{pmatrix}, \text{ respectively.}$$

Since

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 \\ 1/3 & -2/3 & -1/6 \end{pmatrix} = I_3$$

it follows that

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 \\ 1/3 & -2/3 & -1/6 \end{pmatrix} \quad \text{hence}$$

$$A^n = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 \\ 1/3 & -2/3 & -1/6 \end{pmatrix} \quad \text{whence}$$

$$\mathbf{v}_n = A^n \mathbf{v}_0 = \begin{pmatrix} (8\cdot 2^n+(-1)^n-9)/6 \\ x_{n-1} \\ 3 \end{pmatrix},$$

so that, finally,

$$0 \le n \Rightarrow x_n = \frac{8\cdot 2^n+(-1)^n-9}{} \ .$$

Solution by Lin Minghua (China)

From $x_n=x_{n-1}+2x_{n-1}+3$, $n \geq 2$, we get $x_n+x_{n-1}+3=2(x_{n-1}+x_{n-2}+3)=2^{n-1}(x_1+x_0+3)=2^{n+1}$, so

$$x_n-\frac{2^{n+2}}{3}+\frac{3}{2}=-\left(x_{n-1}-\frac{2^{n+1}}{3}+\frac{3}{2}\right)=(-1)^n\left(x_0-\frac{4}{3}+\frac{3}{2}\right)=\frac{(-1)^n}{6},$$

and so

$$x_n=\frac{2^{n+2}}{3}+\frac{(-1)^n}{6}-\frac{3}{2},\ n \geq 0.$$

Also solved by Eric Bahr, Lin Minghua, Massoud Malek, and Grant Morgan