Problem for 2008 October

Proposed by Dan Jurca

Show that there exist infinitely many pairs of integers (x,y) such that

$$x^2+(x+1)^2=y^2.$$

Solution by the proposer

Consider the sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ defined recursively as follows.

$$\begin{aligned} & a_0=3,\ b_0=4,\ c_0=5 \\ 1 \le n \Rightarrow a_n &= a_{n-1}+2b_{n-1}+2c_{n-1} \\ b_n &= 2a_{n-1}+b_{n-1}+2c_{n-1} \\ c_n &= 2a_{n-1}+2b_{n-1}+3c_{n-1} \end{aligned}$$

Obviously $a_n-b_n=-a_{n-1}+b_{n-1}=-(a_{n-1}-b_{n-1})$; and since $a_0-b_0=-1$, it follows (by induction on n) that $0 \le n \Rightarrow a_n-b_n=(-1)^{n-1}$. Next we have $1 \le n \Rightarrow$

$$\begin{aligned} a_n^2 &= a_{n-1}^2+4b_{n-1}^2+4c_{n-1}^2+4a_{n-1}b_{n-1}+8b_{n-1}c_{n-1}+4c_{n-1}a_{n-1}, \\ b_n^2 &= 4a_{n-1}^2+b_{n-1}^2+4c_{n-1}^2+4a_{n-1}b_{n-1}+4b_{n-1}c_{n-1}+8c_{n-1}a_{n-1}, \quad \text{so} \\ a_n^2+b_n^2 &= 5a_{n-1}^2+5b_{n-1}^2+8c_{n-1}^2+8a_{n-1}b_{n-1}+12b_{n-1}c_{n-1}+12c_{n-1}a_{n-1}; \quad \text{and} \\ c_n^2 &= 4a_{n-1}^2+4b_{n-1}^2+9c_{n-1}^2+8a_{n-1}b_{n-1}+12b_{n-1}c_{n-1}+12c_{n-1}a_{n-1}, \quad \text{so} \\ a_n^2+b_n^2-c_n^2 &= a_{n-1}^2+b_{n-1}^2-c_{n-1}^2, \end{aligned}$$

and since $a_0^2+b_0^2-c_0^2=0$, it follows (by induction on n) that $0 \le n \Rightarrow a_n^2+b_n^2=c_n^2$.

Since $0 \le n \Rightarrow 0 < a_n$, $0 < b_n$, and $0 < c_n$, there exist infinitely many pairs of positive integers (x_n,y_n) such that $x_n^2+(x_n+1)^2=y_n^2$, as follows.

$$0 \le n \Rightarrow x_n = \lceil \quad a_n \quad n \equiv 0 \pmod 2$$

$$\begin{cases} b_n & n \equiv 1 \pmod 2, \end{cases}$$

$$y_n = c_n.$$

Solving the above recursion shows that for $0 \leq n$ with integers

$$x_n = \frac{14+5\sqrt{8}}{8}(3+\sqrt{8})^n + \frac{14-5\sqrt{8}}{8}(3-\sqrt{8})^n - \frac{1}{2} \quad \text{and}$$

$$y_n = \frac{20+7\sqrt{8}}{8}(3+\sqrt{8})^n + \frac{20-7\sqrt{8}}{8}(3-\sqrt{8})^n$$

one has $0 \leq n \Rightarrow x_n^2+(x_n+1)^2=y_n^2$.

Also solved by Bojan Basic (Serbia), Lin Minghua (China), Massoud Malek, and Grant Morgan

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