Problem for 2009 April

Communicated by Dan Jurca

Prove the following

Proposition. If x is a number such that $x + \dfrac{1}{x}$ is an integer, then for each integer n it follows that $x^n + \dfrac{1}{x^n}$ is an integer.

Solution by Dan Jurca

We shall prove the slightly more general

Proposition. There exists a sequence $(C_n)_{n=0}^{\infty}$ of functions with the following properties.

- Each C_n is a polynomial with integer coefficients.
- If $x \neq 0$, then $0 \leq n \;\Rightarrow\; x^n + 1/x^n = C_n(x + 1/x)$.

Proof.

Let $C_0(t) = 2,\ C_1(t) = t$, and for $2 \leq n$ let $C_n(t) = tC_{n-1}(t) - C_{n-2}(t)$. Then each C_n is a polynomial of degree n with integer coefficients. (The C_n make an "orthogonal family of polynomials", called by some the Chebyshev polynomials of the second kind.)

If now $x \neq 0$, then $x^0 + 1/x^0 = 1 + 1/1 = 1 + 1 = 2 = C_0(x + 1/x)$, and obviously $x^1 + 1/x^1 = x + 1/x = C_1(x + 1/x)$. Assume now that $2 \leq n$ and $0 \leq k < n \;\Rightarrow\; x^k + 1/x^k = C_k(x + 1/x)$. Then

$$\begin{aligned}
\left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) &= x^n + \frac{1}{x^{n-2}} + x^{n-2} + \frac{1}{x^n} \\
&= \left(x^n + \frac{1}{x^n}\right) + \left(x^{n-2} + \frac{1}{x^{n-2}}\right); \quad \textit{i.e.}, \\
\left(x + \frac{1}{x}\right) C_{n-1}(x + 1/x) &= \left(x^n + \frac{1}{x^n}\right) + C_{n-2}(x + 1/x), \quad \text{so that} \\
x^n + \frac{1}{x^n} &= (x + 1/x)C_{n-1}(x + 1/x) - C_{n-2}(x + 1/x) \\
&= C_n(x + 1/x),
\end{aligned}$$

and the proposition follows.

The statement in the given problem follows at once, since if $x + 1/x \in \mathbf{Z}$, then we have $n \in \mathbf{Z} \;\Rightarrow\; x^n + 1/x^n = C_{|n|}(x + 1/x) \in \mathbf{Z}$. Similarly, if $x + 1/x \in \mathbf{Q}$, then $n \in \mathbf{Z} \;\Rightarrow\; x^n + 1/x^n = C_{|n|}(x + 1/x) \in \mathbf{Q}$.

Also solved by Bojan Bašić (Serbia), Grant Morgan, Àngel Plaza (Spain),
and Jan van Delden (The Netherlands)