Problem for 2009 June

Proposed by Dan Jurca

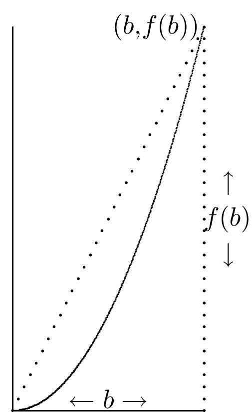
Show that if $0 \le b$, $f : [0, b] \to \mathbf{R}$ is a continuously differentiable function, $f(0) = 0$, and s is the length of the graph of f, then $\sqrt{b^2 + [f(b)]^2} \le s$; and if, further, f is nondecreasing, then also $s \le b + f(b)$, so that

$$\sqrt{b^2 + [f(b)]^2} \le s \le b + f(b).$$

For example, if $0 < p$, $f : [0, \infty) \to \mathbf{R}$ by $f(x) = x^p$, and s_b is the length of the graph of f from $(0, 0)$ to (b, b^p), then

$$\begin{aligned} p < 1 \ &\Rightarrow\ s_b \sim b;\ \textit{i.e.},\ \lim_{b\to\infty} \frac{s_b}{b} = 1 \\ p = 1 \ &\Rightarrow\ s_b = b\sqrt{2} \\ 1 < p \ &\Rightarrow\ s_b \sim b^p;\ \textit{i.e.},\ \lim_{b\to\infty} \frac{s_b}{b^p} = 1. \end{aligned}$$

Solution by the proposer



The lower bound on s is obvious, since $\sqrt{b^2 + [f(b)]^2}$ equals the length of the (straight) line from the point $(0, 0)$ to the point $(b, f(b))$. (Another, but also more tedious, proof appears in the proposition below.)

For the upper bound we observe that if f is differentiable and nondecreasing, then

$$\begin{aligned} 1 + (f')^2 &\le 1 + 2f' + (f')^2 \quad \text{since by hypothesis } 0 \le f' \\ &= (1 + f')^2 \quad \text{so} \\ \sqrt{1 + (f')^2} &\le 1 + f' \quad \text{whence} \\ s = \int_0^b \sqrt{1 + (f')^2} &\le \int_0^b (1 + f') \\ &= b + f(b). \end{aligned}$$

Also solved by Bojan Bašić (Serbia) and John M. Sayer

Proposition. If $0 < b,\ f \in C^1[0,b]$, and $f(0) = 0$, then $\sqrt{b^2 + [f(b)]^2} \le \int_0^b \sqrt{1 + (f')^2}$.

Proof.

First we observe that $0 < x < b \ \Rightarrow\ 0 \le [xf'(x) - f(x)]^2$, from which follows

$$2xf(x)f'(x) \le x^2[f'(x)]^2 + [f(x)]^2. \tag{1}$$

If $\varphi : [0,b] \to \mathbf{R}$ by

$$\varphi(x) = \int_0^x \sqrt{1 + (f')^2} \ - \ \sqrt{x^2 + [f(x)]^2},$$

then φ is differentiable in $(0,b)$ and $0 < x < b \ \Rightarrow$

$$\varphi'(x) = \sqrt{1 + [f'(x)]^2} - \frac{x + f(x)f'(x)}{\sqrt{x^2 + [f(x)]^2}}.$$

If $x + f(x)f'(x) < 0$, then clearly $0 < \varphi'(x)$; otherwise,

$$\begin{aligned}
\varphi'(x) &= \sqrt{1 + [f'(x)]^2} - \frac{\sqrt{[x + f(x)f'(x)]^2}}{\sqrt{x^2 + [f(x)]^2}} \\
&= \frac{\sqrt{x^2 + x^2[f'(x)]^2 + [f(x)]^2 + [f(x)]^2[f'(x)]^2} - \sqrt{x^2 + 2xf(x)f'(x) + [f(x)]^2[f'(x)]^2}}{\sqrt{x^2 + [f(x)]^2}}
\end{aligned}$$

and by (1) the numerator of this fraction is nonnegative. Hence $0 < x < b \ \Rightarrow\ 0 \le \varphi'(x)$, so that φ is nondecreasing. Since $\varphi(0) = 0$ it follows that $x \in [0,b] \ \Rightarrow\ 0 \le \varphi(x)$. In particular $0 \le \varphi(b)$, proving the proposition.