Problem for 2009 October

Proposed by Dan Jurca

Part a. was found on the internet.

Let $(C[0,1], <,>)$ be the (real) inner-product space consisting of all continuous functions $[0,1] \to \mathbf{R}$ with the usual inner-product

$$f, g \in C[0,1] \;\Rightarrow\; <f,g> = \int_0^1 fg = \int_0^1 f(t)g(t)\,dt.$$

a. Let $W_0 = \{f \in C[0,1] \mid f(0) = 0\}$; prove that $W_0^\perp = \{0\}$.

b. Suppose $A \in \mathbf{R}$ and let $S_A = \{f \in C[0,1] \mid \int_0^1 f = A\}$; find $S_A^\perp$.

Solution by the proposer

a. Suppose $g \in C[0,1]$ and $f \in W_0 \;\Rightarrow\; <f,g> = 0$. We show $g = 0$. For suppose $g \neq 0$; then $0 < \|g\|_\infty = \max\{|g(t)| \mid t \in [0,1]\}$, and we derive a contradiction as follows. If $0 < \varepsilon \leq \sqrt{2/3}\,\|g\|_\infty$, $h = 3\varepsilon^2/(2\|g\|_\infty^2)$ — so $0 < h \leq 1$ —, and $f : [0,1] \to \mathbf{R}$ by

$$f(t) = \begin{cases} g(t) \cdot t/h & \text{if } 0 \leq t \leq h \\ g(t) & \text{if } h \leq t \leq 1 \end{cases}$$

then $f \in W_0$, so

$$\begin{aligned}
<f-g, f-g> &= <f,f> -2<f,g> + <g,g> \\
&= \|f\|^2 + \|g\|^2; \quad \text{but also} \\
<f-g, f-g> &= \int_0^1 (f-g)^2 \\
&= \int_0^h (f-g)^2 + \int_h^1 (f-g)^2 \\
&= \int_0^h (f-g)^2 \\
&= \int_0^h [g(t) \cdot t/h - g(t)]^2\,dt \\
&= \frac{1}{h^2} \int_0^h [g(t)]^2 (t-h)^2\,dt \\
&\leq \frac{\|g\|_\infty^2}{h^2} \int_0^h (t-h)^2\,d(t-h) \\
&= \frac{\|g\|_\infty^2}{h^2} \cdot \frac{h^3}{3} \\
&= \|g\|_\infty^2 \cdot h/3 \\
&= \varepsilon^2/2 \\
&< \varepsilon^2.
\end{aligned}$$

Hence $\|f\|^2 + \|g\|^2 < \varepsilon^2$, so $\|g\|^2 < \varepsilon^2$, whence $\|g\| < \varepsilon$. Since ε is an arbitrarily small positive number, it follows that $\|g\| = 0$, and therefore $g = 0$, a contradiction, so $W_0^\perp = \{0\}$.

Remark. With a little more work one can show that if, more generally, $S(x,y) = \{f \in C[0,1] \mid f(x) = y\}$, then $S(x,y)^\perp = \{0\}$. Of course $S(x,y)$ is a subspace of $C[0,1]$ iff $y = 0$.

b. We shall show that $S_0^\perp = \mathbf{R}$, and $A \neq 0 \;\Rightarrow\; S_A^\perp = \{0\}$.

First we show $S_0^\perp = \mathbf{R}$; *i.e.*, $S_0^\perp$ is the subspace of $C[0,1]$ consisting of constant functions. If for some number c we have $t \in [0,1] \;\Rightarrow\; g(t) = c$, and $f \in S_0$, then

$$<f,g> = \int_0^1 fg = c\int_0^1 f = c \cdot 0 = 0,$$

so that $g \in S_0^\perp$. Next, to show that $S_0^\perp$ contains no non-constant function we use the following lemma and corollary.

Lemma. If $f \in C[0,1]$ and

$$\left(\int_0^1 f\right)^2 = \int_0^1 f^2$$

then f is a constant function.

Proof.

With $A = \int_0^1 f$, if $\int_0^1 f^2 = \left(\int_0^1 f\right)^2 = A^2$, then

$$\int_0^1 (f-A)^2 = \int_0^1 (f^2 - 2Af + A^2) = \int_0^1 f^2 - 2A\int_0^1 f + A^2 = \int_0^1 f^2 - 2A^2 + A^2 = \int_0^1 f^2 - \left(\int_0^1 f\right)^2 = 0.$$

Since $(f-A)^2$ is nonnegative and continuous, it follows that $(f-A)^2 = 0$. Hence $f = A$, a constant.

Corollary. If $g \in C[0,1]$ and g is a non-constant function, then there exists $f \in C[0,1]$ such that

$$\int_0^1 f = 0 \quad \text{and} \quad \int_0^1 fg \neq 0.$$

Proof.

Let $\int_0^1 g = A$ and consider $f = g - A$. Then $\int_0^1 f = \int_0^1 (g-A) = 0$, so $g - A = f \in S_0$; also

$$\int_0^1 fg = \int_0^1 (g-A)g = \int_0^1 g^2 - \int_0^1 Ag = \int_0^1 g^2 - A\int_0^1 g = \int_0^1 g^2 - A^2 = \int_0^1 g^2 - \left(\int_0^1 g\right)^2 \neq 0$$

by the lemma, since g is not constant.

It follows that if $g \in C[0,1]$ and g is not a constant function, then there exists $f \in S_0$ such that $<f,g> \neq 0$. Therefore $g \notin S_0^\perp$, so that $S_0^\perp$ consists precisely of all constant functions; *i.e.*, $S_0^\perp = \mathbf{R}$.

Now suppose $A \neq 0$, $g \in C[0,1]$, and $g \neq 0$; we find $f \in S_A$ such that $<f,g> \neq 0$. For if

$$f = \begin{cases} g + A & \text{if } \int_0^1 g = 0 \\ A & \text{if } \int_0^1 g \neq 0 \end{cases} \qquad \text{then } \int_0^1 f = A, \text{ so } f \in S_A; \text{ however}$$

$$<f,g> = \begin{cases} \int_0^1 (g+A)g = \int_0^1 g^2 + A\int_0^1 g = \int_0^1 g^2 \neq 0 \text{ since } 0 \leq g^2 \neq 0 & \text{if } \int_0^1 g = 0 \\ \int_0^1 Ag = A\int_0^1 g \neq 0 \text{ since } A \neq 0 & \text{if } \int_0^1 g \neq 0. \end{cases}$$

Therefore $g \notin S_A^\perp$, so that $A \neq 0 \;\Rightarrow\; S_A^\perp = \{0\}$.

Also solved by Bojan Bašić (Serbia)