Problem for 2010 July

Proposed by Dan Jurca

Suppose $1 \le k \le n \;\Rightarrow\; x_k \in \mathbf{R}$,

$$1 \le i \le n,\; 1 \le j \le n \;\Rightarrow a_{ij} = \begin{cases} x_i x_j & \text{if } i \ne j \\ x_i x_j + 1 = x_i^2 + 1 & \text{if } i = j, \end{cases}$$

and consider the $n \times n$ matrix $A = (a_{ij})$.

a. Find the determinant of A.

b. Find the characteristic equation and the spectrum of A.

c. Find n linearly independent eigenvectors of A.

d. Find the inverse of A.

Solution by the proposer

If each $x_k = 0$, then $A = I_n$, the $n \times n$ identity matrix, so the problem is trivial. Therefore, suppose $x_m \ne 0$ for some m, $1 \le m \le n$. If $b_{ij} = x_i x_j$ and $B = (b_{ij}) = A - I_n$ then the spectrum of A, $\sigma(A)$, equals $\{1 + \lambda \mid \lambda \in \sigma(B)\}$. With $s = x_1^2 + x_2^2 + \cdots + x_n^2$ we will show that the characteristic equation of B is $q(\lambda) = (\lambda - s)\lambda^{n-1}$. Consider the vectors $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^n$ as follows.

$$\mathbf{v}^m = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad k \ne m \;\Rightarrow\; \mathbf{v}^k = \begin{bmatrix} v_1^k \\ v_2^k \\ v_3^k \\ \vdots \\ v_n^k \end{bmatrix} \text{ where } v_i^k = \begin{cases} -x_k & \text{if } i = m \\ x_m & \text{if } i = k \\ 0 & \text{if } i \ne m \text{ and } i \ne k \end{cases}$$

We compute $B\mathbf{v}^m$ and each $B\mathbf{v}^k$, $k \ne m$.

$$(B\mathbf{v}^m)_i = \sum_{j=1}^n b_{ij} x_j = \sum_{j=1}^n (x_i x_j) x_j = \sum_{j=1}^n x_i (x_j x_j) = x_i \sum_{j=1}^n x_j^2 = s x_i \quad \text{whence}$$
$$B\mathbf{v}^m = s\mathbf{v}^m; \quad \text{and}$$
$$k \ne m \;\Rightarrow\; (B\mathbf{v}^k)_i = \sum_{j=1}^n b_{ij} v_j^k = b_{im} v_m^k + b_{ik} v_k^k = (x_i x_m) \cdot -x_k + (x_i x_k) \cdot x_m = 0 \quad \text{whence}$$
$$B\mathbf{v}^k = \mathbf{0} = 0\mathbf{v}^k.$$

It follows that $\sigma(B) = \{s; 0\}$, and associated eigenvectors are $\mathbf{v}^m;\ \mathbf{v}^1, \ldots, \mathbf{v}^{m-1}, \mathbf{v}^{m+1}, \ldots, \mathbf{v}^n$, which are linearly independent. (It is obvious that $\{\mathbf{v}^k \mid k \ne m\}$ is a linearly independent set. Since $k \ne m \Rightarrow A\mathbf{v}^k = 0\mathbf{v}^k$ and $A\mathbf{v}^m = s\mathbf{v}^m \ne 0\mathbf{v}^m$, it follows that $\mathbf{v}^m$ is not a linear combination of the other $\mathbf{v}^k$.) Hence $\sigma(A) = \{s+1; 1\}$ and associated eigenvectors are $\mathbf{v}^m;\ \mathbf{v}^1, \ldots, \mathbf{v}^{m-1}, \mathbf{v}^{m+1}, \ldots, \mathbf{v}^n$; $\det(A) = s+1$; and the characteristic polynomial of A is $p(\lambda) = (\lambda - (s+1))(\lambda - 1)^{n-1}$. Since $\det(A) = s + 1 \ne 0$, A is invertible.

Next let

$$c_{ij} = \begin{cases} s + 1 - x_i^2 & \text{if } i = j \\ -x_i x_j & \text{if } i \ne j, \end{cases}$$

and let $C = (c_{ij})$; we show that $AC = \det(A) \cdot I_n$.

First we compute

$$
\begin{aligned}
(AC)_{ii} &= \sum_{k=1}^{n} a_{ik}c_{ki} \\
&= \sum_{k=1}^{i-1} a_{ik}c_{ki} + a_{ii}c_{ii} + \sum_{k=i+1}^{n} a_{ik}c_{ki} \\
&= \sum_{k=1}^{i-1}(x_i x_k)\cdot(-x_k x_i) + (x_i^2+1)\cdot(s+1-x_i^2) + \sum_{k=i+1}^{n}(x_i x_k)\cdot(-x_k x_i) \\
&= -x_i^2\sum_{k=1}^{i-1} x_k^2 + sx_i^2 + x_i^2 - x_i^4 + s + 1 - x_i^2 - x_i^2 \sum_{k=i+1}^{n} x_k^2 \\
&= -x_i^2\sum_{k=1}^{n} x_k^2 + sx_i^2 + s + 1 \\
&= -x_i^2 s + sx_i^2 + s + 1 \\
&= s+1; \quad \text{and then} \\
i<j \;\Rightarrow\; (AC)_{ij} &= \sum_{k=1}^{n} a_{ik}c_{kj} \\
&= \sum_{k=1}^{i-1} a_{ik}c_{kj} + a_{ii}c_{ij} + \sum_{k=i+1}^{j-1} a_{ik}c_{kj} + a_{ij}c_{jj} + \sum_{k=j+1}^{n} a_{ik}c_{kj} \\
&= \sum_{k=1}^{i-1}(x_i x_k)\cdot(-x_k x_j) + (x_i^2+1)\cdot(-x_i x_j) + \sum_{k=i+1}^{j-1}(x_i x_k)\cdot(-x_k x_j) + \\
&\qquad + (x_i x_j)\cdot(s+1-x_j^2) + \sum_{k=j+1}^{n}(x_i x_k)\cdot(-x_k x_j) \\
&= -x_i x_j \sum_{k=1}^{i-1} x_k^2 - x_i x_j \sum_{k=i+1}^{j-1} x_k^2 - x_i x_j \sum_{k=j+1}^{n} x_k^2 - x_i^3 x_j - x_i x_j + sx_i x_j + x_i x_j - x_i x_j^3 \\
&= -x_i x_j \left[s - x_i^2 - x_j^2\right] - x_i^3 x_j + sx_i x_j - x_i x_j^3 \\
&= -x_i x_j \left[s - x_i^2 - x_j^2 + x_i^2 - s + x_j^2\right] \\
&= 0;
\end{aligned}
$$

and similarly $j<i \;\Rightarrow\; (AC)_{ij}=0$. Therefore $AC=(s+1)\cdot I_n = \det A \cdot I_n$, so that, finally,

$$
A^{-1} = \frac{1}{s+1}C, \quad \text{where}
$$
$$
s = x_1^2 + \cdots + x_n^2.
$$

Remark. If we let $\mathbf{x}=\mathbf{v}^m=(x_1,x_2,\ldots,x_n)^T$, then we have $A=\mathbf{x}\mathbf{x}^T+I_n$ and $C=(s+2)I_n-A$. Hence, for example, $A\mathbf{x}=(\mathbf{x}\mathbf{x}^T+I_n)\mathbf{x}=\mathbf{x}(\mathbf{x}^T\mathbf{x})+\mathbf{x}=\mathbf{x}s+\mathbf{x}=(s+1)\mathbf{x}$, so that $s+1\in\sigma(A)$ with associated eigenvector $\mathbf{x}$. Next $AC=A\left((s+2)I_n-A\right)=(s+2)A-A^2=sA+I_n-I_n+2A-A^2=sA+I_n-(A-I_n)^2=sA+I_n-\mathbf{x}\mathbf{x}^T\mathbf{x}\mathbf{x}^T=sA+I_n-\mathbf{x}s\mathbf{x}^T=sA+I_n-s\mathbf{x}\mathbf{x}^T=sA+I_n-s(A-I_n)=(s+1)I_n$, so that $A^{-1}=1/(s+1)\cdot C=1/(s+1)\cdot((s+2)I_n-A)$, as before.

Also solved by Matthew Felix and Massoud Malek. Massoud Malek has generalized this result.