Problem for 2011 February

Communicated by Dan Jurca

Show that of all n-gons inscribed in the unit circle the regular n-gon has greatest area.

Solution by Dan Jurca

If the central angles subtended by the sides of an inscribed n-gon are $\theta_1,\ \theta_2,\ \ldots,\ \theta_n$, then the area of the polygon equals

$$\frac{1}{2}\sum_{i=1}^{n}\sin\theta_i,$$

and clearly $1 \le i \le n \ \Rightarrow\ 0 < \theta_i < \pi$. Since the function $f : [0,\pi] \to \mathbf{R}$ by $f(\theta) = \sin\theta$ is concave it follows from Jensen's inequality that

$$\begin{aligned}\frac{\sum_{i=1}^{n}\sin\theta_i}{n} &\le \sin\frac{\sum_{i=1}^{n}\theta_i}{n}\\ &= \sin\frac{2\pi}{n}, \quad \text{so that}\\ \frac{1}{2}\sum_{i=1}^{n}\sin\theta_i &\le \frac{n}{2}\sin\frac{2\pi}{n},\end{aligned}$$

and this last quantity equals the area of a regular n-gon inscribed in the unit circle.

Also solved by Matthew Felix, Massoud Malek, Bill Nico, John M. Sayer, and Jan van Delden (the Netherlands)