Problem for 2011 May

Communicated by Dan Jurca

The following is well-known.

Suppose $3 \leq n$ and a regular n-gon is inscribed in the unit circle in the x-y plane with one vertex at the point $P = (1{,}0)$, and diagonals are drawn from P to each of the other $n-1$ vertices; show that the product of the lengths of these diagonals equals n.

Solution by several solvers

Let $\zeta = e^{2\pi i/n}$, a primitive n-th root of unity; then the vertices of the regular n-gon inscribed in the unit circle are at the points $1 = \zeta^0,\ \zeta^1,\ \zeta^2, \ldots,\ \zeta^{n-1}$, and the problem is to find

$$|1-\zeta| \times |1-\zeta^2| \times \cdots \times |1-\zeta^{n-1}|.$$

One can factor the polynomial $z^n - 1$ in the following two ways.

$$\begin{aligned} z^n - 1 &= (z-1)(z^{n-1} + z^{n-2} + \cdots + 1) \\ &= (z-1)(z-\zeta)(z-\zeta^2)\cdots(z-\zeta^{n-1}) \end{aligned}$$

Therefore for each $z \in \mathbf{C}$ we have

$$(z-\zeta)(z-\zeta^2)\cdots(z-\zeta^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1.$$

Setting $z = 1$ in this last equation and taking absolute values of the two sides we find

$$|1-\zeta| \times |1-\zeta^2| \times \cdots \times |1-\zeta^{n-1}| = n,$$

as desired.

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