Problem for 2012 June

Proposed by Kourosh Ghaderi

Prove that if G is a group with only finitely many subgroups, then G is of finite order.

Solution by Dan Jurca

We consider two possibilities, as follows.

i. $g \in G \Rightarrow$ the order of g is finite; or

ii. there exists $g \in G$ such that the order of g is infinite.

In the first case, since $G = \cup\{\langle g \rangle \mid g \in G\}$, where $\langle g \rangle$ is the cyclic group generated by g, and there exist only finitely many such $\langle g \rangle$, it follows that (the underlying set) G is the finite union of finite sets, so that G is finite.

In the second case, suppose $g \in G$ and the order of g is infinite. Then the cyclic group $\langle g \rangle$ generated by g is isomorphic to the group $\mathbf{Z}$, the (abelian) group of integers under addition. (An isomorphism is defined by extending $g \mapsto 1$.) But there exist infinitely many subgroups of $\mathbf{Z}$. For if m and n are distinct positive integers, then the quotient groups $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$ and $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ contain m and n elements, respectively. Therefore G does not contain an element of infinite order.

Also solved by Massoud Malek and Winston Teitler