Problem for 2012 October

Communicated by Dan Jurca

a. Prove that if $1 \le n \ \Rightarrow a_n \in \mathbf{R},\ (a_n)_{n=1}^{\infty} \to L$, and

$$1 \le n \ \Rightarrow \ x_n = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n},$$

then $(x_n)_{n=1}^{\infty} \to L$.

b. Prove that if $1 \le n \ \Rightarrow \ 0 < a_n \in \mathbf{R},\ (a_n)_{n=1}^{\infty} \to L$, and

$$1 \le n \ \Rightarrow \ x_n = \sqrt[n]{a_1 a_2 a_3 \cdots a_n},$$

then $(x_n)_{n=1}^{\infty} \to L$.

Solution by Dan Jurca

a. Since the sequence $(L - a_n)_{n=1}^{\infty}$ converges (to 0) it is bounded; let $M = \max\{|L - a_n| \mid 1 \le n\}$. If $0 < \epsilon$ there exists n_0 such that $n_0 \le n \ \Rightarrow \ |L - a_n| < \epsilon/2$. Hence if

$$\max\left\{n_0, \frac{(2M - \epsilon)n_0}{\epsilon}\right\} < n, \quad \text{then}$$

$$\begin{aligned}
|L - x_n| &= \left| L - \frac{a_1 + a_2 + \cdots + a_n}{n} \right| \\
&= \frac{1}{n}|nL - (a_1 + a_2 + \cdots + a_n)| \\
&= \frac{1}{n}|(L - a_1) + (L - a_2) + \cdots + (L - a_n)| \\
&\le \frac{1}{n}\sum_{i=1}^{n}|L - a_i| \\
&= \frac{1}{n}\left(\sum_{i=1}^{n_0}|L - a_i| + \sum_{i=n_0+1}^{n}|L - a_i|\right) \\
&< \frac{1}{n}\left(n_0 M + (n - n_0)\frac{\epsilon}{2}\right) \\
&= \frac{1}{2n}(2Mn_0 - \epsilon n_0 + n\epsilon) \\
&= \frac{1}{2n}[(2M - \epsilon)n_0 + n\epsilon] \\
&= \frac{\epsilon}{2n}\left[\frac{(2M - \epsilon)n_0}{\epsilon}\right] + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2n} \cdot n + \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

so that $(x_n)_{n=1}^{\infty} \to L$.

b. Since $0 < a_n$ for each n, one easily sees (and can prove) that $0 \le L$. If $0 < L$, then by convergence of $(a_n)_{n=1}^{\infty}$ and continuity of the logarithm function $(\log a_n)_{n=1}^{\infty} \to \log L$. Hence by part a. above

$$\left(\frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n}\right)_{n=1}^{\infty} \to \log L,$$

so that $\left(\log(a_1 a_2 \cdots a_n)^{1/n}\right)_{n=1}^{\infty} \to \log L$, so $\left((a_1 a_2 \cdots a_n)^{1/n}\right)_{n=1}^{\infty} \to L$; *i.e.*, $(x_n)_{n=1}^{\infty} \to L$.
Now suppose $L = 0$, $0 < \epsilon$, and $n_0 \le n \ \Rightarrow\ 0 < a_n < \epsilon/2$. Let $M = \max\{a_n \mid 1 \le n\}$. Then $0 < M$ and since for each (strictly) positive real number α

$$\lim_{n\to\infty} \alpha^{1/n} = 1,$$

there exists n_1 such that

$$n_1 \le n \ \Rightarrow\ \left[\left(\frac{2M}{\epsilon}\right)^{n_0}\right]^{1/n} < 2.$$

Then $\max\{n_0, n_1\} < n \ \Rightarrow$

$$\begin{aligned}
0 < x_n &= (a_1 a_2 \cdots a_n)^{1/n} \\
&= \left[(a_1 a_2 \cdots a_{n_0})(a_{n_0+1} a_{n_0+2} \cdots a_n)\right]^{1/n} \\
&< \left[M^{n_0} \cdot \left(\frac{\epsilon}{2}\right)^{n-n_0}\right]^{1/n} \\
&= \left[\left(\frac{2M}{\epsilon}\right)^{n_0} \cdot \left(\frac{\epsilon}{2}\right)^{n}\right]^{1/n} \\
&= \left[\left(\frac{2M}{\epsilon}\right)^{n_0}\right]^{1/n} \cdot \frac{\epsilon}{2} \\
&< 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

so that $(x_n)_{n=1}^{\infty} \to 0$.

Also solved by Massoud Malek and John M. Sayer