Problem for 2013 June

Proposed by Dan Jurca

Find a non-constant continuous function $f : \mathbf{R} \to \mathbf{R}$ such that

$$x \in \mathbf{R} \;\Rightarrow\; \int_0^{f(x)} f = f(x); \qquad i.e.,$$

$$\int_0^{f(x)} f(t)\,dt = f(x).$$

Solution by the proposer

Assuming differentiability of f we find by the (corollary to the) fundamental theorem of calculus and the chain rule

$$\frac{d}{dx}\int_0^{f(x)} f = \frac{d}{dx} f(x) \qquad \text{so}$$

$$f(f(x)) \cdot f'(x) = f'(x).$$

Hence either $f'(x) = 0$ or $f(f(x)) = 1$. This computation suggests we look for functions f such that $f \circ f = 1$. Hence suppose $\varphi : [0, \infty) \to \mathbf{R}$ such that the following conditions hold.

- φ is continuous;
- $\varphi(0) = 0$;
- $-1 \le \varphi$;
- $\exists x \in [0, \infty)$ such that $\varphi(x) \ne 0$.

Then we define $f : \mathbf{R} \to \mathbf{R}$ as follows.

$$x \in \mathbf{R} \;\Rightarrow\; f(x) = \begin{cases} 1 + \varphi(-x) & \text{if } x \le 0 \\ 1 & \text{if } 0 \le x \end{cases}$$

Then $f : \mathbf{R} \to \mathbf{R}$ is non-constant and continuous (and $x \in \mathbf{R} \;\Rightarrow\; f(f(x)) = 1$), and

$$x \le 0 \;\Rightarrow\; \int_0^{f(x)} f = \int_0^{1+\varphi(-x)} f = \int_0^{1+\varphi(-x)} 1 = 1 + \varphi(-x) = f(x), \quad \text{and}$$

$$0 \le x \;\Rightarrow\; \int_0^{f(x)} f = \int_0^1 f = \int_0^1 1 = 1 = f(x),$$

so that for any such φ the function f will do. For example, (with $\varphi(x) = x^2$)

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \le 0 \\ 1 & \text{if } 0 \le x. \end{cases}$$

This f is differentiable.