Problem for 2013 December

Proposed by Dan Jurca

What is an indefinite integral of the function $f : (0, \pi/2) \to \mathbf{R}$ by

$$f(x) = \frac{\cot x}{\ln \sin x}$$

and why isn't it?

Solution by the proposer

One is tempted to write

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{\cot x}{\ln \sin x}\,dx \\
&= \int \frac{1}{\ln \sin x} \cdot \frac{1}{\sin x} \cdot \cos x\,dx \\
&= \int \frac{1}{\ln \sin x} \cdot d(\ln \sin x) \\
&= \ln \ln \sin x + C, \quad \text{and (checking)} \\
\frac{d}{dx}(\ln \ln \sin x) &= \frac{1}{\ln \sin x} \cdot \frac{d}{dx} \ln \sin x \\
&= \frac{1}{\ln \sin x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x \\
&= \frac{1}{\ln \sin x} \cdot \frac{1}{\sin x} \cdot \cos x \\
&= \frac{\cot x}{\ln \sin x},
\end{aligned}$$

except that the "function" defined by the expression "$\ln \ln \sin x$" does not exist.

However, if $F : (0, \pi/2) \to \mathbf{R}$ by $F(x) = \ln |\ln \sin x|$, then

$$\begin{aligned}
F'(x) &= \frac{1}{\ln \sin x} \cdot \frac{d}{dx} \ln \sin x \\
&= \frac{1}{\ln \sin x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x \\
&= \frac{1}{\ln \sin x} \cdot \frac{1}{\sin x} \cdot \cos x \\
&= \frac{1}{\ln \sin x} \cdot \frac{\cos x}{\sin x} \\
&= \frac{1}{\ln \sin x} \cdot \cot x \\
&= \frac{\cot x}{\ln \sin x} \\
&= f(x).
\end{aligned}$$

Also solved by Arthur Fabian and Amit Singh