Problem for 2014 March

Communicated by Dan Jurca

Suppose $1 \leq n,\ A = (a_{ij})$ is an $n \times n$ matrix of integers, β is an integer, and if the rows of A represent "integers in base β"(*i.e.*, we require neither $1 \leq \beta$ nor $1 \leq i, j \leq n \Rightarrow 0 \leq a_{ij} < \beta$) then each row of A represents an integer divisible by (the nonzero integer) m. Show that m divides the determinant of A.

For example, if

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & -3 & 10 \\ 3 & -6 & -3 \end{bmatrix},$$

then using base 7 we have

$$\begin{aligned}
\text{row1} &= (1)(2)(6)_7 = (1) \times 7^2 + (2) \times 7 + (6) \times 1 = 49_{10} + 14_{10} + 6 = 69_{10} = 3 \times 23_{10} \\
\text{row2} &= (2)(-3)(10)_7 = (2) \times 7^2 + (-3) \times 7 + (10) \times 1 = 98_{10} - 21_{10} + 10_{10} = 87_{10} = 3 \times 29_{10} \\
\text{row3} &= (3)(-6)(-3)_7 = (3) \times 7^2 + (-6) \times 7 + (-3) \times 1 = 147_{10} - 42_{10} - 3 = 102_{10} = 3 \times 34_{10},
\end{aligned}$$

so that each row is (in base 7) a multiple of 3; and $\det A = 123_{10} = 3 \times 41_{10}$, also a multiple of 3.

Solution by Dan Jurca

We interpret the problem statement as meaning that there exists a nonnegative integer m such that

$$1 \leq i \leq n \ \Rightarrow\ \exists q_i \in \mathbf{Z} \text{ such that } a_{i,1}\beta^{n-1} + a_{i,2}\beta^{n-2} + \cdots + a_{i,n-1}\beta + a_{i,n} = mq_i.$$

Let $B = (b_{ij})$ be the $n \times n$ matrix as follows.

$$1 \leq i \leq n,\ 1 \leq j \leq n \ \Rightarrow\ b_{ij} = \begin{cases} 0 & \text{if } j \neq i \text{ and } j \neq n \\ 1 & \text{if } j = i \\ \beta^{n-i} & \text{if } j = n \end{cases}$$

Then B is an upper-triangular matrix each diagonal entry of which equals 1, so that $\det B = 1$. We compute the entry $(AB)_{in}$ in row i and column n of the product AB as follows.

$$\begin{aligned}
1 \leq i \leq n \ \Rightarrow\ (AB)_{in} &= \sum_{k=1}^{n} a_{i,k}b_{k,n} \\
&= \sum_{k=1}^{n} a_{i,k}\beta^{n-k} \\
&= a_{i,1}\beta^{n-1} + a_{i,2}\beta^{n-2} + \cdots + a_{i,n-1}\beta + a_{i,n} \\
&= mq_i
\end{aligned}$$

Since each entry in column n of AB equals a multiple of m, it follows that m divides $\det(AB) = \det A \cdot \det B = \det A \cdot 1$, so that m divides the determinant of A.

Also solved by Athur Fabian and Massoud Malek