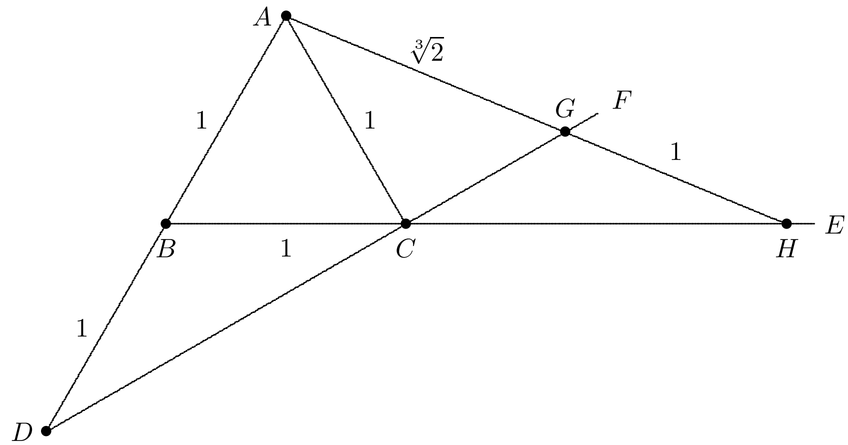
Problem for 2014 April

Communicated by Dan Jurca

The following diagram is a copy of one which appears on page 44 in an article by Fernando Q. Gouvêa in the vol. 34 no. 1 issue of *MAA Focus* magazine, February/March 2014. Prove that the diagram is correct.

That is, if equilateral triangle ABC is constructed with each side of length 1, the line through A and B is extended to D so that the length of segment BD equals 1, the line through D and C is drawn and extended, and a line through A is drawn through a point H on line BC in such a way that the length of segment GH equals 1 (where G is the point where line DC intersects line AH), then the length of segment AG equals $\sqrt[3]{2}$.



Solution by Dan Jurca

If the diagram is drawn in the rectangular Cartesian plane so that point C is at the origin of coordinates and point B is at $(-1, 0)$, then A is at $(-1/2, \sqrt{3}/2)$, and D is at $(-3/2, -\sqrt{3}/2)$. Suppose H is at $(u, 0)$ and G is at (v, w). Then the slope of line DF equals $(\sqrt{3}/2)/(3/2) = 1/\sqrt{3}$, so that an equation of line DF is $y = x/\sqrt{3}$. Next, the slope of line AH equals $-(\sqrt{3}/2)/(u + 1/2) = -\sqrt{3}/(2u + 1)$, so that an equation of line AH is $y = -\sqrt{3}/(2u + 1) \cdot (x - u)$. These lines intersect at point G with coordinates (v, w); hence

$$\frac{v}{\sqrt{3}} = -\frac{\sqrt{3}}{2u+1}(v - u), \text{ so } (2u + 1)v = 3(u - v), \text{ and } (2u + 4)v = 3u.$$

Therefore

$$v = \frac{3u}{2u+4} \quad \text{and} \quad w = \frac{\sqrt{3}u}{2u+4}.$$

Since the length of segment GH equals 1, we have $(v - u)^2 + (w - 0)^2 = 1$, so

$$\left(\frac{3u}{2u+4} - u\right)^2 + \left(\frac{\sqrt{3}u}{2u+4} - 0\right)^2 = 1, \text{ whence } (-2u^2 - u)^2 + 3u^2 = (2u + 4)^2.$$

Therefore $4u^4 + 4u^3 + 4u^2 = 4u^2 + 16u + 16$, so $u^4 + u^3 = 4u + 4$, and $u^3(u + 1) = 4(u + 1)$, so $u = \sqrt[3]{4}$. Hence

$$\begin{aligned}
(\text{length of } AH)^2 &= (u + 1/2)^2 + (\sqrt{3}/2)^2 \\
&= u^2 + u + 1 \\
&= (\sqrt[3]{4})^2 + \sqrt[3]{4} + 1 \\
&= \sqrt[3]{4} + \sqrt[3]{16} + 1 \\
&= (\sqrt[3]{2})^2 + 2\sqrt[3]{2} + 1 \\
&= (\sqrt[3]{2} + 1)^2, \\
\text{so length of } AH &= \sqrt[3]{2} + 1.
\end{aligned}$$

Therefore length of $AG = (\text{length of } AH) - 1 = (\sqrt[3]{2} + 1) - 1 = \sqrt[3]{2}$.